LEBESGUE DENSITY AS A SET FUNCTION

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Lebesgue (or metric) density is usually considered as a point function in the sense that a fixed subset of a space X is given and then the value of the density of this set is obtained at various points of the space. Suppose the density is considered in another sense. That is, let a point x of the space be fixed and consider the class $\mathscr{D}(x)$ of all sets whose density exists at this point. Then to each set E in $\mathscr{D}(x)$ we assign the value of its density at x, and denote this number by $D_x(E)$. Thus from this point of view the density is a finite set function. It was shown in [2] that if the space X is the real line then the image of $\mathscr{D}(x)$ under D_x is the closed unit interval.

It is evident from the definition of density of sets of real numbers, which we give below, that D_x is a finitely additive, subtractive, monotone, nonnegative set function and the class $\mathscr{D}(x)$ is closed under the formation of complements, proper differences, and disjoint unions. Therefore, if $\mathscr{D}(x)$ were closed under the formation of intersections, D_x would be a finitely additive measure. This however is not the case for if

$$R_n = \left\{x\colon \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right) < x < \frac{1}{n}\right\},$$

$$L_n = \left\{x\colon -\frac{1}{n} < x < -\frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right)\right\}$$

and

$$L_n^* = \left\{x\colon -\frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right) < x < -\frac{1}{n+1}\right\},$$

the sets $\bigcup_n (R_n \cup L_n) = E$ and $\bigcup_n (R_n \cup L_n^*) = F$ are members of $D(0)$ but $E \cap F$ is not. In fact $D_0(E) = D_0(F) = \frac{1}{2}$ and the upper density of $E \cap F$ at zero is not less than $\frac{1}{2}$ while the lower density of $E \cap F$ at zero is zero.

In part 1 of this note we prove a theorem which is somewhat of an analogue of the Lebesgue density theorem [3] in the following respect. As noted above D_x is not a finitely additive measure, but we show that the upper density at x, $\bar{D}_x$, is a finitely subadditive outer measure defined on the class of all Lebesgue measurable subsets of X and the class of $\bar{D}_x$-measurable sets is the class of all sets whose density exists at x and has the value zero or one. In part 2 a Lebesgue density of a measurable set E on a fixed F_σ set of measure zero is defined and a similar result

Received May 24, 1960. Presented to American Mathematical Society. Part 1 of this note constitutes a portion of the author's doctoral dissertation written at Iowa State College under the direction of Professor H. P. Thielman.