DISSIPATIVE OPERATORS IN A BANACH SPACE

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1. Introduction. The Hilbert space theory of dissipative operators[1] was motivated by the Cauchy problem for systems of hyperbolic partial differential equations (see [5]), where a consideration of the energy of, say, an electromagnetic field leads to an L_2 measure as the natural norm for the wave equation. However there are many interesting initial value problems in the theory of partial differential equations whose natural setting is not a Hilbert space, but rather a Banach space. Thus for the heat equation the natural measure is the supremum of the temperature whereas in the case of the diffusion equation the natural measure is the total mass given by an L_1 norm. In the present paper a suitable extension of the theory of dissipative operators to arbitrary Banach spaces is initiated.

An operator A with domain $\mathfrak{D}(A)$ contained in a Hilbert space H is called dissipative if

$$re(Ax, x) \leqq 0 , \qquad x \in \mathfrak{D}(A) , \tag{1.1}$$

and maximal dissipative if it is not the proper restriction of any other dissipative operator. As shown in [5] the maximal dissipative operators with dense domains precisely define the class of generators of strongly continuous semi-groups of contraction operators (i.e. bounded operators of norm $\leqq 1$). In the case of the wave equation this furnishes us with a description of all solutions to the Cauchy problem for which the energy is nonincreasing in time. Our aim will be to characterize the generators of all strongly continuous semigroups of contraction operators in an arbitrary Banch space. For this we shall use the notion of a semi inner-product, introduced in [4].

DEFINITION 1.1. *A semi inner-product is defined on a complex (real) vector space $\mathfrak{X}$ if to each pair x, y in $\mathfrak{X}$ there corresponds a complex (real) number $[x, y]$ in such a way that:*

$$\begin{aligned} [x + y, z] &= [x, z] + [y, z] \\ [\lambda x, y] &= \lambda[x, y] \end{aligned} \qquad x, y, z \in \mathfrak{X} \text{ and } \lambda \text{ complex (real)} ; \tag{1.2}$$

$$[x, x] > 0 \qquad \text{for } x \neq 0 ; \tag{1.3}$$

$$| [x, y] |^2 \leqq [x, x] [y, y] \qquad x, y \in \mathfrak{X} . \tag{1.4}$$

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[1] The term operator will be understood throughout as denoting a linear transformation, not necessarily bounded, with domain and range subspaces of the same space.