ON PLANE CURVES WITH CURVATURE

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As a temporary abbreviation we say that a (planar) curve is an *R-curve* provided that its curvature is continuous and does not exceed $1/R$. A well-known theorem of Schwartz [1 page 63, 2, 6, 7] states that if two points on a circle C of radius R are joined by any R-curve, X, then the arc length of X does not exceed the length of the smaller arc of C unless indeed the length of X is at least as great as that of the larger arc of C determined by the two points.

In this paper we call attention to an area of largely unexplored mathematics, of which the above theorem of Schwartz can be taken as a takeoff point. Here, we only begin the exploration, raise some questions that we hope will prove stimulating, and invite others to discover the proofs of the definitive theorems, proofs that have eluded us. Roughly, the principal question is: given two points (in the Euclidean plane) what kind of R-curve can connect them? One approach towards making this question precise is as follows: Focus attention on two R-curves that connect the two given points and ask under what circumstances is it possible to gradually deform the first curve into the second, where at each stage of the deformation the curve is an R-curve connecting the two given points. Actually, our investigation is primarily concerned with a related problem in which the two given curves, and every intermediary curve, have the same tangent direction at the first of the two points, as well as at the second. In this way we become interested in the arc components of a space of curves. This leads to similarities and connections with the work of Graustein-Whitney and Smale [8, 9]. However the curvature restriction leads to new problems.

The idea for a curvature constraint arises naturally from considerations of a particle that moves at constant speed and subject to a maximum possible force. If that particle leaves a certain point heading in a certain direction and desires to arrive at another point from a certain direction what are the paths available to the particle? If it tries to take a certain available path but through errors does not quite traverse this path what is the nature of the possible neighboring paths? (Homotopy classes.) These questions represent the background for this paper.

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