TAME CANTOR SETS IN E^3

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1. Introduction. Let M_1 be the straight line interval $[0, 1]$, M_2 be the sum of $[0, 1/3]$ and $[2/3, 1]$ obtained by deleting the open middle third of M_1, M_3 be the sum of the four intervals obtained by deleting from M_2 the open middle thirds of the components of $M_2, \cdots$. The intersection M of $M_1, M_2, \cdots$ is the familiar Cantor set. Any homeomorphic image of M is called a *Cantor set*.

It may be shown that if C_1 is a Cantor set on the real line, there is a homeomorphism of the line onto itself taking C_1 onto M. Also, if C_2 is a Cantor set in a plane, there is a homeomorphism of the plane onto itself taking C_2 into a line. For these reasons, we say that each Cantor set in a line or a plane is tame. A Cantor set C_3 in E^3 is *tame* if there is a homeomorphism of E^3 onto itself taking C_3 into a line. Not each Cantor set in E^3 is tame. Antoine gives an example of such a wild Cantor set in [1]. A diagram of one is found in [3, 8]. Blankenship has described [7] wild Cantor sets in Euclidean spaces of all dimensions greater than 2.

Characterizations of tame Cantor sets are provided by Theorems 1.1, 3.1, 4.1 and 5.1. In § 6 we prove theorems about the sums of tame Cantor sets and apply these results in § 7 to show that for each closed 2-dimensional set X in E^3, there is a homeomorphism h of E^3 onto itself that is close to the identity and such that $h(X)$ contains no straight line interval. An example is given of a disk containing intervals pointing in all directions showing that such a homeomorphism h may need to be something more than a rigid motion.

A *simple neighborhood* in E^3 is an open set topologically equivalent to the interior of a sphere. By using simple neighborhoods, we give the following characterization of tame Cantor sets. The proof is obtained in a straight forward fashion.

THEOREM 1.1. *A necessary and sufficient condition that a Cantor set C in E^3 be tame is that for each positive number ε, C can be covered by a finite collection of mutually exclusive simple neighborhoods of mesh less than ε.*

Proof. That the condition is necessary follows from the facts that a homeomorphism of a closed and bounded set in E^3 is uniformly continuous and if the homeomorphic image of C lies on a line, this image

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