ON THE GRAPH STRUCTURE OF CONVEX POLYHEDRA IN n-SPACE

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1. Introduction. The contents of this paper arose from work done in developing an algorithm for finding all vertices of convex polyhedral sets defined by systems of linear inequalities [1]. The following natural questions were raised: if we consider the vertices of convex polyhedral sets as the points, and the edges as the lines of a graph, does there exist a path or a cycle which goes through all points exactly once (i.e., does there exist a Hamiltonian path or cycle)? The answer to both questions is negative: there exists, in general, no Hamiltonian path or cycle. A simple example of a convex polyhedral set in 3-space whose graph contains no Hamiltonian path (and hence no Hamiltonian cycle) has recently been devised by T. A. Brown [2]. The classic example of Tutte [7] shows only that no Hamiltonian cycle exists.

In this paper, however, we show that such graphs do have the general property of being n-tuply connected. According to Whitney's Theorem [8] this implies that there exist n disjoint paths between any pair of vertices. We give a new proof of this fact based on an application of the Max-Flow Min-Cut Theorem [3], [5]. Finally, we point out that all proofs are based on the theory of linear programming, and thus on theory which itself rests on the properties of convex polyhedral sets.

2. The result. *A graph* $G(\pi, \Delta)$ is defined to be a finite collection of points π together with a collection of lines Δ. The lines consist of pairs of distinct points and Δ is thus some given subset of the collection of all possible lines formed from points in π. A line (p_1, p_2) is said to be *incident* to each of the points p_1 and p_2. A point is said to have *degree* n if n lines are incident to it. A *path* is a collection of lines $(p_1, p_2), (p_2, p_3), \cdots, (p_k, p_{k+1})$ with $p_i \neq p_j$ $j = i + 1$, and $k \geqq 1$. Paths are said to be *disjoint* if they have no points except possibly first and last points in common. A *cycle* is a path with $k \geqq 2$ whose first and last points are the same. We say a graph G is *connected* if there exists a path between any two of its points. We define an *n-tuply connected* graph G to be a graph with at least $n + 1$ points and such that it is impossible to disconnect it by dropping out $n - 1$ or fewer points.

Consider the polyhedral convex set S in n-space described by the system of linear inequalities

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