# THE ANALYTIC-FUNCTIONAL CALCULUS IN COMMUTATIVE TOPOLOGICAL ALGEBRAS

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**1. Introduction.** The idea of an analytic-functional calculus involving holomorphic functions $f$ of several variables seems to have originated with Shilov [6]. Shilov uses Weil's integral formula [8] to construct, for each $f$ holomorphic on the joint spectrum of elements $a_1, \cdots, a_n$ of a commutative Banach algebra $A$, an element $b$ of that algebra, deserving the name $f(a_1, \cdots, a_n)$ because of the function $b$ yields on the space of maximal ideals. Shilov's requirement that $a_1, \cdots, a_n$ generate the algebra was removed in [1]. Waelbrock [8], perhaps idependently of [6], treated the general case and indeed that of more general algebras. Waelbrock uses the ordinary form of Cauchy's integral, but also deeper ideal-theoretic results of K. Oka and H. Cartan. He shows moreover that one can arrange the mapping $f \to f(a_1, \cdots, a_n)$ so as to be an algebra-homomorphism, which is not obvious for the method of Shilov-Arens-Calderón [6, 1]. One purpose of the present paper is to show that this results from that method also. Another is to give a careful exposition of the Weil integral, or rather a weaker but more effective form involving integration on affine rather that analytic polyhedra. Although we have elsewhere sketched a proof of such a result, we dealt only with $n = 2$, as Weil did, and there was some question about the combinatorial procedure in the general case.

We desired to establish also a covariance property of the functional calculus (see 4.2 below) which enables us (see § 5) to extend the functional calculus to certain inverse limits of Banach algebras.

However, the most interesting discovery is that one can just as well deal with holomorphic $A$-valued functions $f$, rather than merely complex-valued functions. (For a trivial example, if $f(\lambda) = \lambda a$ on the spectrum of $b$, then $f(b) = ab$.) The attractive thing is that by extending the technique in this way, the distinction between the case in which $a_1, \cdots, a_n$ generate $A$, and that in which they do not, simply does not arise, nor does the matter of polynomial-convexity which was the great discovery of, but at the same time the indispensable tool for, Shilov.

The actual integral representation for functions holomorphic in the usual sense, on a suitably convex, compact subset of $C^n$ is then *derived* from the theorem (4.1, 4.4 below) concerning the case of $A$-valued functions.

**2. Holomorphic differential forms with values in a topological**

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