AXIOMS FOR NON-RELATIVISTIC QUANTUM MECHANICS

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Introduction. In the approach to the axiomatization of quantum mechanics of George W. Mackey [7], a series of plausible axioms is completed by a final axiom that is more or less *ad hoc*. This axiom states that a certain partially ordered set—the set P of all two-valued observables—is isomorphic to the lattice of all closed subspaces of Hilbert space. The question arises as to whether this axiom can be deduced from others of a more *a priori* nature, or, more generally, whether the lattice of closed subspaces of Hilbert space can be characterized in a physically meaningful way. Our central result is a characterization of this lattice which may serve as a step in the indicated direction, although there is not now a precise sense in which our axioms are more plausible than his. Its principal features may be described as follows.

Suppose that P is an atomic lattice, define an element to be *finite* if it is the join of a finite number of points, and suppose that the unit element is not finite, but is the join of a countable set of points. Suppose for the moment that

(F) The lattice under every finite element of P is a real (or complex) projective geometry.

Then one additional axiom, which appears to be particularly mild from an operational viewpoint, is sufficient and necessary for us to show that P is isomorphic to the lattice of closed subspaces of a separable, infinite dimensional real (or complex) Hilbert space.

Of course, (F) is not taken as an axiom, but is deduced from more primitive assumptions. This part of the development follows well-known lines, but the structure of P (and its set S of states) permits us to give it a rather simple form. For example, in order to conclude that the lattice under every finite element of P is a projective geometry, we need make, in addition to the atomicity of P, only the following three assumptions: P is not a Boolean algebra; the lattices under any pair of finite elements of the same dimension are isomorphic; a certain weak (and rather intuitive) form of the modular law holds under finite elements (Theorem 2.1).

In a preliminary chapter we examine the interrelation of a number of regularity properties which a pair P, S satisfying a slight refinement of Mackey's basic axioms might have, and show that a few of the more

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