AUTOMORPHISMS OF SEPARABLE ALGEBRAS

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1. Introduction. In this note we begin by noticing that for any commutative ring C, the isomorphism classes of finitely generated, projective C-modules of rank one (for the definition, see § 2) form an abelian group $\mathscr{J}(C)$ which reduces to the ordinary ideal class group if C is a Dedekind domain. In [2], Auslander and Goldman proved that if $\mathscr{J}(C)$ contains only one element then every automorphism of every central separable C-algebra is inner. Using similar techniques, we prove that for general C and for any central separable C-algebra A, $\mathscr{J}(C)$ contains a subgroup isomorphic to the group of automorphisms of A modulo inner ones. We characterize both this subgroup and the factor group. For example, in the case of an integral domain or a noetherian ring, the subgroup is the set of classes of projective ideals in C which become principal in A (i.e., $\operatorname{Ker}\beta$ in Theorem 7). If C is a Dedekind ring and A is the (split) algebra of endomorphisms of a projective C-module of rank n, the subgroup is the set of classes of ideals whose nth powers are principal.

2. Generalization of the ideal class group. Let C be a commutative ring[1] and let J be a projective C-module. Then for every maximal ideal M in C, the module[2] $J \otimes C_M$ is a projective, hence free, C_M-module. Following [7, § 3] we say J has rank one if for all M, $J \otimes C_M$ is free on one generator,[3] i.e. $J \otimes C_M \cong C_M$ as C_M-modules.

DEFINITION. $\mathscr{J}(C)$ will denote the set of isomorphism classes of finitely generated, projective, rank one C-modules. If J is a finitely generated, projective, rank one C-module, $\{J\}$ will denote the isomorphism class of J.

We note that if $\{J\} \in \mathscr{J}(C)$ then J is faithful, for if an ideal I annihilates J then $0 = I(J \otimes C_M) \cong IC_M \cong I \otimes C_M$ for every M, and so $I = 0$ [4, Chap. VII, Ex. 11].

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[1] All rings will be assumed to have units, all modules will be unitary, and if R is a subring of S then R will contain the unit element of S. A homomorphism of rings will preserve unit elements.

[2] The unadorned $\otimes$ always means tensor product over C. C_M denotes the ring of quotients of C with respect to the maximal ideal M.

[3] $J \otimes C_M \cong C_M$ for all M does not imply that J is either finitely generated or projective. For example, let C be the ring of integers and $J = \bigcup_n Cp_1^{-1} \cdots p_n^{-1}$ where p_i is the ith prime.