ON THE DIFFERENCE AND SUM OF BASIC SETS OF POLYNOMIALS

N. N. MIKHAIL AND M. NASSIF

1. If the two functions $f(z)$ and $g(z)$ are connected by the relation

$$g(z+1) - g(z) = f(z),$$

then $f(z)$ is called the difference of $g(z)$ and $g(z)$ is the sum of $f(z)$. These relations are denoted by

$$f(z) = \Delta g(z) \; ; \qquad g(z) = \mathscr{S} f(z) \; ,$$

and it is obvious that any function of period unity can be added to the sum of a given function. The authors considered recently [1] the difference set $\{u_n(z)\}$ and the sum set $\{v_n(z)\}$ of a given simple set of polynomials[1] $\{p_n(z)\}$. These sets are the simple sets defined by

$$u_n(z) = \Delta p_{n+1}(z) \; ; \qquad (n \geqq 0) \; , \tag{1.1}$$

$$v_0(z) = 1; \; v_n(z) = \mathscr{S} p_{n-1}(z) \; ; \qquad (n \geqq 1) \; , \tag{1.2}$$

and the indetermination in the sum set is removed by supposing that

$$v_n(0) = 0 \; ; \qquad (n \geqq 1) \; . \tag{1.3}$$

The main result of the above mentioned work concerns the order δ and σ of the difference and sum sets respectively of a simple set of a given order ω. In fact, it has been shown that

$$\delta \leqq \max(1, \omega) \; , \tag{1.4}$$

and

$$\sigma \leqq \omega + 1 \; . \tag{1.5}$$

Our aim in the present paper is to generalise these results for more general classes of basic sets of polynomials. It will be here shown that, with suitable modification of the definition of difference sets, the upper bound in (1.4) remains the same for the most general classes of basic sets of polynomials. As for the sum sets, it will be here proved that, in order to get a finite upper bound for the order of the sum set, a limitation on the class of basic sets is inevitable.

2. This section and the following one are devoted to the study of

Received January 9, 1960, and in revised form May 23, 1960.

[1] The reader is supposed to be acquainted with the the theory of basic sets of polynomials as given by Whittaker [3].