GROUP MEMBERSHIP IN RINGS AND SEMIGROUPS

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1. Introduction. Let R be a semigroup or associative ring. A group G in R is a subset of R which is a group under the multiplication on R. That is, G contains an idempotent e which acts as a multiplicative identity on G and if $\alpha \in G$ then there exists an element $\alpha' \in G$ such that $\alpha\alpha' = \alpha'\alpha = e$. An element α of R is said to be a group element in R if α belongs to some group in R.

The problem of deciding whether a given element of R is a group element has been investigated in various types of rings in [3], [4], [5], [6], [10], [11]. The purpose of the present paper is the generalization and extension of some results of Barnes and Schneider [3], Drazin [4] and Farahat and Mirsky [6].

Section 2 of this paper extends some results of [6] on the imbedding of the groups contained in a ring with identity in the group of units of the ring.

In § 3 use is made of the concept of left π-regularity. McCoy [9] introduced the concept of π-regularity, the consequences of which have developed in [1], [2], and [8]. It imposes a finitness condition satisfied, for example, by rings with minimum condition, by nil rings, by the "divided" rings of [6] and by direct sums of such rings. This condition is found to be sufficient in many of the cases where [6] uses the condition that the ring be a direct sum of divided rings. Moreover, the condition of left π-regularity is applicable to the case of semigroups. Under this condition, it is shown that if S is an extension of a semigroup or ring R, $\alpha \in R$ and α is a group element in S, then α is a group element of R.

Section 4 deals with conditions under which some power of a given element of R is a group element.

Section 5 gives a necessary and sufficient condition for the same property in terms of annihilators.

In order to point up the comparative weakness of the condition of left π-regularity of a ring necessary and sufficient conditions are given in § 6 that a left π-regular ring be a direct sum of divided rings.

2. Groups in rings with identity. Throughout this section R will denote a ring with an identity element 1 and U will denote the group of units of R.

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