# TWO CLASSES OF DIOPHANTINE EQUATIONS

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**1. Introduction.** It is easily shown, by means of the Thue-Siegel-Roth theorem, that the equation

$$(1) \qquad x^2 + 7M^2 = N^y \ , \qquad M \geqq 1 \ , \quad N \geqq 2$$

has only a finite number of nonnegative integral solutions $x, y$. Here we show that this equation cannot have more than $56MN$ integral solutions, and if $(M, 7N) = 1$, it cannot have more than $28N$ solutions. While these are reasonable upper bounds for the number of solutions, they are almost certainly excessively large. With appropriate assumptions on $M$ and $N$ we are able to obtain much smaller bounds.

Previously, Nagell [8] and others [9], [5] showed that the equation $x^2 + 7 = 2^y$ has exactly five solutions. We show that when $N$ is an odd integer the equation $x^2 + 7 = N^y$ has at most two solutions and it has no solution unless $N$ is an odd power of a prime.

In the second part of this paper we determine an upper bound, in terms of $M$ and $n$, on the number of primitive solutions (i.e. $x$ and $y$ coprime) of the equation

$$(2) \qquad x^2 + 7M^2 = y^n \ , \qquad M \geqq 1 \quad \text{and} \quad n \geqq 3 \ .$$

In making our calculations, we make use of results which were obtained by means of a $p$-adic argument.

Nagell has shown that the equation $x^2 + D = y^n$ has at most one solution when $D = 2$ or $8$ [7] and has no solution when $D$ is a square free integer congruent to 1 or 2 (modulo 4) [6]. Ljungren [3], [4] has shown that this equation has at most one solution when $D = p^2$ ($p$ a prime) or $D = 1 + 2^s t$ ($s$ and $t$ odd integers $\geqq 3$.)

**2. Notation.** Let $K = Q(\sqrt{-7})$, where $Q$ is the field of rational numbers. Let $\mathfrak{O}$ be the set of algebraic integers in $K$, then $\mathfrak{O}$ is a unique factorization ring having $\pm 1$ as its only units. Let $\mathfrak{S}$ denote the set of rational primes which split in $\mathfrak{O}$ into the product of two non-associate conjugate primes. $\mathfrak{S}$ does not contain $7 = (\sqrt{-7})(-\sqrt{-7})$. Let $\mathfrak{I}$ be the set of rational primes not in $\mathfrak{S} \cup (7)$, then all integers in $\mathfrak{I}$ are primes in $\mathfrak{O}$.

Set $\rho = \frac{1}{2}(1 + \sqrt{-7})$, then $2 = \rho \cdot \rho'$ and $\rho$ is a root of the equation $z^2 - z + 2 = 0$. If $p$ is in $\mathfrak{S}$, we write $p = \pi_p \cdot \pi'_p$.

If $p$ is a rational prime and $m$ is a rational integer, we write $\| m \|_p$ for the largest rational integer $s$ such that $p^s \mid m$.

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