PRINCIPLES OF PARTIAL REFLECTION IN THE SET THEORIES OF ZERMELO AND ACKERMANN

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It was shown in [4] that the Zermelo-Fraenkel set theory may be obtained by adjoining to the Zermelo theory $\boldsymbol{Z}$ an axiom schema called the principle of complete reflection. (This schema, denoted here by $\boldsymbol{CR}$, and other notions involved in these introductory remarks will be described explicitly later.) A schema of partial reflection, called here $\boldsymbol{PR}$, which is also valid in $\boldsymbol{ZF}$ was described in [5]. As we shall see in § 2, the theory $\boldsymbol{T}_0 = \boldsymbol{Z} + \boldsymbol{PR}$ is a very strong one, in which, apparently, all of the 'ordinary' set-theoretical constructions can be carried out.

Nevertheless, we shall show in § 4 that $\boldsymbol{T}_0$ is much weaker than $\boldsymbol{ZF}$. Indeed, we shall prove within $\boldsymbol{ZF}$ the existence of numerous well-behaved models of a theory $\boldsymbol{T}_1$ which deals with classes as well as sets and is based on a partial reflection principle $\boldsymbol{PR}_1$ stronger than $\boldsymbol{PR}$.

The theory $\boldsymbol{A}^*$ of [3], which is the set theory of Ackermann [1] with an axiom of regularity added, deals with classes as well as set (members of V). Regarding its 'purely set-theoretical' part $\boldsymbol{A}^*/V$, it is known that $\boldsymbol{A}^*/V \subseteqq \boldsymbol{ZF}$ (cf. [3]). An interesting question, which remains open, is whether $\boldsymbol{A}^*/V$ and $\boldsymbol{ZF}$ coincide. The development of the theory $\boldsymbol{A}^*$ has already been carried out in [1] and [3] to a considerable extent. In particular, it was shown in [3] that $\boldsymbol{T}_0 \subseteqq \boldsymbol{A}^*/V$ and, moreover, that another sort of partial reflection principle, called here $\boldsymbol{R}$, is valid in $\boldsymbol{A}^*$. As we shall observe below, the central axiom schema (γ) of $\boldsymbol{A}^*$ may, in fact, be replaced by $\boldsymbol{R}$.

In order to investigate the strength of $\boldsymbol{A}^*$, we shall, in § 6, establish within $\boldsymbol{A}^*$ the same facts concerning the existence of models of $\boldsymbol{T}_1$, which have been previously proved in § 4 on the basis of $\boldsymbol{ZF}$. The very short proofs within $\boldsymbol{ZF}$ of these facts depend heavily on the axiom of replacement, or that of complete reflection. In order to establish the same facts within $\boldsymbol{A}^*$ we are forced to carry further the general development of $\boldsymbol{A}^*$. In particular various results will be obtained within $\boldsymbol{A}^*$ concerning proper classes, which have hitherto been little investigated.

§ 1. Preliminaries. Each set theory $\boldsymbol{T}$ we shall consider is formalized in the first order logic with identity and has as its only non-logical symbols the binary predicate ε plus, in some cases, the individual constant V. Its only sentential connective is | (joint denial) and its only quantifier is $\exists$. (The symbols '$\wedge$', '$\forall$', etc., used in the metalanguage

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