IMBEDDING COMPACT RIEMANN SURFACES IN 3-SPACE

TILLA KLOTZ

1. Any sufficiently smooth surface in E^3 has a conformal structure imposed upon it by the metric of the surrounding space. If there is a conformal homeomorphism between a Riemann surface and some C^k surface in E^3, then the Riemann surface is said to be C^k imbedded in E^3. We deal below with some aspects of the problem of C^∞ imbedding compact Riemann surfaces in E^3.

Since every compact Riemann surface of genus zero is conformally equivalent to the sphere, the problem becomes non-trivial only when genus $g \geqq 1$. Recently Garsia and Rodemich [4] proved that every compact Riemann surface of genus 1 can be C^∞ imbedded in E^3. We therefore restrict our attention compact Riemann surfaces of genus $g \geqq 2$.

2. Before stating the main result, we recall some definitions. For each fixed genus $g \geqq 2$, choose a fixed compact Riemann surface R_g of genus g. Then a marked Riemann surface of genus g is an equivalence class

$$\mathscr{S} = \langle (R, \alpha) \rangle$$

of pairs, where R is a compact Riemann surface of genus g, and α is a homotopy class of orientation preserving topological mappings of R_g onto R. The equivalence

$$(R, \alpha) \sim (R', \alpha')$$

holds if and only if R and R' are conformally equivalent under a homeomorphism in the homotopy class $\alpha^{-1}\alpha'$. A marked Riemann surface is said to be C^k imbedded in E^3 if the first member of some representative pair is C^k imbedded in E^3.

It is well known (see, for example, [1]) that the set of all marked Riemann surfaces of genus g may be made into a metric space in a natural manner, thereby becoming the Teichmüller space T_g. We define $\Sigma_g \subset T_g$ to be the set of all $\mathscr{S} \in T_g$ which can be C^∞ imbedded in E^3. Note that Σ_g is never empty.

But then, the conjecture that every compact Riemann surface of genus $g \geqq 2$ is C^∞ imbeddable in E^3 is equivalent to the conjecture that Σ_g is both open and closed in T_g.[1] In what follows we deal exclusively

Received June 15, 1960. Part of this research was completed while the author was a National Science Foundation Fellow.

[1] It is in this form that the problem was suggested to the author by Professor Lipman Bers, to whom we express our gratitude.