MOMENT INEQUALITIES OF PÓLYA FREQUENCY FUNCTIONS

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1. Introduction and summary. The theory of totally positive kernels and Pólya frequency functions has been applied fruitfully in several domains of mathematics and statistics. In this paper we derive moment inequalities governing Pólya frequency functions of various orders. In particular, Pólya frequency functions on the positive axis are characterized in terms of inequalities on the moments.

We begin by introducing the necessary definitions and notation, and we also review some of the fundamental background. A function f(x, y)of two real variables, x ranging over X and y ranging over Y, is said to be totally positive of order k (TP_k) if for all $x_1 < x_2 < \cdots < x_m$, $y_1 < y_2 < \cdots < y_m$ $(x_i \in X, y_j \in Y)$, and all $1 \leq m \leq k$,

(1)
$$f\begin{pmatrix}x_1, x_2, \cdots, x_m\\y_1, y_2, \cdots, y_m\end{pmatrix} = \begin{vmatrix}f(x_1, y_1) & f(x_1, y_2) & \cdots & f(x_1, y_m)\\f(x_2, y_1) & f(x_2, y_2) & \cdots & f(x_2, y_m)\\\vdots & \vdots & \vdots\\f(x_m, y_1) & f(x_m, y_2) & \cdots & f(x_m, y_m)\end{vmatrix} \ge 0$$

Typically, X is an interval of the real line, or a countable set of discrete values on the real line such as the set of all integers or the set of nonnegative integers; similarly for Y. When X or Y is a set of integers, we use the term "sequence" rather than "function".

We record for later reference the following consequence of (1) proved in [3], p. 284.

If f(x, y) is TP_k where X and Y represent open intervals on the line and all the indicated derivatives exist, then

(2)
$$\begin{vmatrix} f(x,y) & \frac{\partial}{\partial y}f(x,y) & \cdots & \frac{\partial^{n-1}}{\partial y^{n-1}}f(x,y) \\ \frac{\partial}{\partial x}f(x,y) & \frac{\partial^2}{\partial x\partial y}f(x,y) & \cdots & \frac{\partial^n}{\partial x\partial y^{n-1}}f(x,y) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^{n-1}}{\partial x^{n-1}}f(x,y) & \frac{\partial^n}{\partial x^{n-1}\partial y}f(x,y) & \cdots & \frac{\partial^{2n-2}}{\partial x^{n-1}\partial y^{n-1}}f(x,y) \\ \end{vmatrix} \ge 0$$

A related concept to total positivity is that of sign reverse regularity.

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