CONTINUITY AND CONVEXITY OF PROJECTIONS AND
BARYCENTRIC COORDINATES IN CONVEX POLYHEDRA

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If \( s_0, \ldots, s_n \) are linearly independent points of real \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) then each point \( x \) of their convex hull \( S \) has a (unique) representation \( x = \sum_{i=0}^{n} \lambda_i(x)s_i \) with \( \lambda_i(x) \geq 0 \) (\( i = 0, \ldots, n \)) and \( \sum_{i=0}^{n} \lambda_i(x) = 1 \), and the barycentric coordinates \( \lambda_0, \ldots, \lambda_n \) are continuous convex functions on \( S \) (cf. [3, p. 288]). We shall show in this paper that given any finite set \( s_0, \ldots, s_m \) of points of \( \mathbb{R}^n \) we can assign barycentric coordinates \( \lambda_0, \ldots, \lambda_m \) to their convex hull \( S \) in such a way that each coordinate is continuous on \( S \) and that one prescribed coordinate (\( \lambda_0 \) say) is convex on \( S \) (Theorem 2); the author does not know whether it is always possible to make all the coordinates convex simultaneously (cf. Example 3). In proving Theorem 2 we shall use certain "projections" which we now define; these projections are in general distinct from those of [1, p. 614] and [2, p. 12]. Given two distinct points \( s_0 \) and \( s \) of \( \mathbb{R}^n \), let \( s_0s \) be the open half-line consisting of all points \( s_0 + \lambda(s - s_0) \) with \( \lambda > 0 \); given a point \( s_0 \) of \( \mathbb{R}^n \) and a closed subset \( S \) of \( \mathbb{R}^n \) such that \( s_0 \notin S \), let \( C(s_0, S) \) be the "cone" formed by the union of all open half-lines \( s_0s \) with \( s \) in \( S \); and given a point \( x \) in such a cone \( C(s_0, S) \), let \( \pi(x) \) be the (unique) point of \( s_0x \cap S \) which is closest to \( s_0 \). Then we shall call the function \( \pi \) the "projection of \( C(s_0, S) \) on \( S \)." Our proof of Theorem 2 depends on the fact that if \( S \) is a convex polyhedron then \( \pi \) is continuous (Theorem 1). This result may appear to be obvious, but it is not immediately obvious how a formal proof should be given; moreover, as we shall show (Examples 1 and 2), the conclusion need not remain true for polyhedra \( S \) which are not convex or for convex sets \( S \) which are not polyhedra. The author is indebted to the referee for improvements to Lemma 3, Example 1, and Example 2, and for the remark at the end of § 1.

1. Projections. For any subset \( A \) of \( \mathbb{R}^n \) we shall denote by \( H(A) \) the convex hull of \( A \) and by \( L(A) \) the affine subspace of \( \mathbb{R}^n \) spanned by \( A \) (cf. [2, pp. 21, 15]). If \( A = \{s_1, \ldots, s_p\} \) we shall write \( H(A) = H(s_1, \ldots, s_p) \) and \( L(A) = L(s_1, \ldots, s_p) \). Given two points \( x \) and \( y \) of \( \mathbb{R}^n \) we shall denote by \( (x, y) \) the inner product of \( x \) and \( y \) and by \( |x - y| \) the Euclidean distance \( \sqrt{(x - y, x - y)} \) between \( x \) and \( y \).

**Lemma 1.** Let \( s_0 \) be a point of \( \mathbb{R}^n \), let \( S \) be a closed convex subset of \( \mathbb{R}^n \) such that \( s_0 \notin S \), and let \( \pi \) be the projection of \( C(s_0, S) \) on \( S \). Suppose that points \( x, s_1, \ldots, s_p \) of \( S \) and real numbers \( \lambda_1, \ldots, \lambda_p \) are...