THE GROUP OF AUTOMORPHISMS OF THE HOLOMORPH OF A GROUP

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1. Introduction. If $G = HK$ where H is a normal subgroup of the group G and where K is a subgroup of G with the trivial intersection with H, then G is said to be a *semi-direct product* of H by K or a *splitting extension* of H by K. We can consider a splitting extension G as an ordered triple $(H, K; \phi)$ where ϕ is a homomorphism of K into the automorphism group $\mathfrak{A}(H)$ of H. The ordered triple $(H, K; \phi)$ is the totality of all ordered pairs (h, k), $h \in H$, $k \in K$, with the multiplication

$$(h, k)(h', k') = (h\phi_k(h'), kk') .$$

If ϕ is a monomorphism of K into $\mathfrak{A}(H)$, then $(H, K; \phi)$ is isomorphic to $(H, \phi(K); \iota)$ where ι is the identity mapping of $\phi(K)$, and therefore G is the *relative holomorph* of H with respect to a subgroup $\phi(K)$ of $\mathfrak{A}(H)$. If ϕ is an isomorphism of K onto $\mathfrak{A}(H)$, then G is the *holomorph* of H.

Let H be a group, and let G be the holomorph of H. We are considering H as a subgroup of G in the usual way. Gol'fand [1] studied the group $\mathfrak{A}_H(G)$ of automorphisms of G each of which maps H onto itself, the group $\mathfrak{J}(G)$ of inner automorphisms of G, and the factor group $\mathfrak{A}_H(G)/\mathfrak{J}(G)$. In case H is abelian, this factor group is isomorphic to the first cohomology group of $\mathfrak{A}(H)$ acting on H, as Mills [4] mentioned. In § 2, we generalize Gol'fand's results by dealing with a relative holomorph instead of with the holomorph. The fact that ϕ is a monomorphism is essential for the proof. Hence this generalization of Gol'fand's theory is in some sense the best possible one. Gol'fand [1], Miller [3], Mills [4], Peremans [5] and Specht [6, pp. 101–102] discussed the group of automorphisms of the holomorph of some groups. In § 3 and 4, we discuss the group of automorphisms of the holomorph of some other uncomplicated groups. As applications we can describe the group of automorphisms of the holomorph of symmetric groups and the group of automorphisms of the holomorph of subgroups of the additive group of rational numbers.

We set up our basic device which determines all automorphisms of a splitting extension $G = (H, K; \phi)$ in terms of mappings of H and K. It also enables us to compute the product of two automorphisms of G.

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