REAL COMMUTATIVE SEMIGROUPS ON THE PLANE

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A *real semigroup* is a topological semigroup containing a sub-semigroup R isomorphic to the multiplicative semigroup of real numbers, embedded so that 1 is an identity and 0 is a zero. This paper is devoted to a preliminary study of real commutative semigroups on the plane and especially to characterizing the product semigroup on $R \times R$. It leans heavily on the fundamental paper [5] of Mostert and Shields which in turn depends the paper [1] of Faucett who, among other things, characterized the multiplicative semigroup on the closed unit interval. Characterizations of the multiplicative semigroup of nonnegative real numbers and of all real numbers were given in [4] and [3] respectively. (In connection with the latter characterization, also see [2]).

Nothing like a complete description of all real commutative semigroups on the plane can be given at this time, even under the additional hypothesis that there are no (non-zero) nilpotent elements. A crude classification can be given however on the basis of the number and arrangement of the components of the maximal subgroup $H(1)$. If $H(1)$ is connected then the semigroup is necessarily the multiplicative semigroup of complex numbers. If $H(1)$ is not connected then the component G of the identity in $H(1)$ is always isomorphic to the two dimensional vector group. There can be precisely two components in $H(1)$; in this case, $H(1)$ may be dense or not. There are at least two instances of the former (see Examples 1 and 2 of § 6) and at least two instances of the latter (see Examples 3 and 4 of § 6). If there are more than two components, but there are no nilpotent elements, then the number of components of $H(1)$ is four and $H(1)/G$ is isomorphic to the four group. Example 5 of § 6 shows that even in this case $H(1)$ need not be dense and the suggestion is that there are many instances of this case. A characterization of the product semigroup on $R \times R$ appears in § 5.

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Preliminaries. The closure of a subset A of a topological space is denoted A^-. The set-theoretic difference of two sets A and B is denoted by $A \backslash B$.

A binary operation, or multiplication, is denoted by juxtaposition. By a semigroup S we mean a topological semigroup, that is, a Hausdorff space with a continuous associative multiplication. All semigroups

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