PHYSICAL INTERPRETATION AND STRENGTHENING OF M. H. PROTTER'S METHOD FOR VIBRATING NONHOMOGENEOUS MEMBRANES; ITS ANALOGUE FOR SCHRÖDINGER'S EQUATION

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The origin of this work lies partly in *M. H. Protter*'s method [7], [8], partly in two papers [3], [5], developing the idea, found in *Payne-Weinberger* [6], of auxiliary one-dimensional problems; the physical interpretation in § 3 rejoins that of [2] and [4].

1. We consider the first eigenvalue λ_1 of a nonhomogeneous membrane with specific mass $\rho(x, y) \geqq 0$ covering a plane domain D and elastically supported (elastic coefficient $k(s)$) along its boundary Γ:

$$\Delta u + \lambda\rho(x, y)u = 0 \text{ in } D\,, \quad \frac{\partial u}{\partial n} + k(s)u = 0 \text{ along } \Gamma\,,$$

where $\vec{n}$ is the outward normal.

Every continuous and piecewise smooth function $v(x, y)$ furnishes an upper bound for λ_1: By Rayleigh's principle

$$\lambda_1 = \mathrm{Min}_v \frac{D(v) + \oint_\Gamma k(s)v^2 ds}{\iint_D \rho v^2 dA}\,,$$

where ds is the length element, dA the element of area, and $D(v)$ the Dirichlet integral $\iint_D \mathrm{grad}^2\, v\, dA$. *The Minimum is realized if* $v = u_1(x, y)$ (*first eigenfunction*, satisfying $\Delta u_1 + \lambda_1\rho u_1 = 0$).

In the opposite direction, we are here in search of a Maximum principle for λ_1, from which we could calculate lower bounds.

2. Let us consider in D a sufficiently regular vector field $\vec{p}$ (we shall discuss presently what discontinuities are allowed), satisfying the *condition*

$$\vec{p} \cdot \vec{n} \leqq k(s) \qquad \text{along } \Gamma\,. \tag{1}$$

$$\begin{aligned}\mathrm{grad}^2\, u_1 + (\vec{p}^2 - \mathrm{div}\, \vec{p})\, u_1^2 &= -\mathrm{div}\, (u_1^2\vec{p}) + \mathrm{grad}^2\, u_1 + u_1^2\vec{p}^2 + 2u_1\, \mathrm{grad}\, u_1 \cdot \vec{p} \\ &= -\mathrm{div}(u_1^2\vec{p}) + (\mathrm{grad}\, u_1 + u_1\vec{p})^2 \geqq -\mathrm{div}(u_1^2\vec{p})\,.\end{aligned}$$

Let us integrate this inequality:

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