COMPLETE HOLOMORPHS

JOHN HARVEY

1. Introduction. Throughout this paper let G be an additive group, and denote the group of all automorphisms of G by $A(G)$ and the holomorph of G by $K(G)$. Then $K(G) = A(G) \times G$, where $(\alpha, a) + (\beta, b) = (\alpha\beta, a\beta + b)$ for all elements (α, a) and (β, b) of $K(G)$. We prove that if G is abelian and $x \rightarrow 2x$ is an automorphism of G, then $K(G)$ is complete if and only if $G' = 1 \times G$ is a characteristic subgroup of $K(G)$. From this it follows that if G is abelian, $x \rightarrow 2x$ is an automorphism of G, and $A(G)$ is abelian, then $K(G)$ is complete.

In § 3 we derive analogous results for ordered abelian groups. Then we show that any divisible, torsion free, abelian group can be ordered so that its o-holomorph is o-complete. It is known (see [2]) that the holomorph of a non-abelian group is not complete. In § 4 we give an example of a non-abelian o-group with an o-complete o-holomorph. Finally, we show that the lexicographically ordered direct sum of two o-complete groups is again o-complete.

2. Complete holomorphs. Recall that a group is *complete* if it has a trivial center and all of its automorphisms are inner.

In 1957, W. Peremans [3] investigated under what conditions the holomorph of an abelian group is complete. He was able to derive a necessary and sufficient condition for the holomorph to be complete when $x \rightarrow 2x$ is an automorphism of the group. Using this result he was then able to prove that if $x \rightarrow 2x$ is an automorphism of the group and if the group is either directly indecomposable, the direct sum of cyclic groups, or is divisible, then the holomorph is complete.

We derive a necessary and sufficient condition which is simpler in statement than that of Peremans. However, before this theorem can be proved some preliminary lemmas are necessary which have independent interest. Let B be a subgroup of $A(G)$, and let τ be a mapping from B into G. Then τ is a *crossed homomorphism* if for all α and β in B,

$$(\alpha\beta)\tau = (\alpha\tau)\beta + \beta\tau .$$

LEMMA 2.1. *Let G be an abelian group. If τ is a crossed homomorphism of $A(G)$ into G, then the mapping χ of $K(G)$ into itself defined by*

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