OSCILLATION CRITERIA FOR THIRD-ORDER LINEAR DIFFERENTIAL EQUATIONS

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1. Introduction. This paper is concerned with the oscillatory properties of the third-order linear differential equation

$$y''' + P(x)y'' + Q(x)y' + R(x)y = 0 \tag{1.1}$$

where $P(x), Q(x)$, and $R(x)$ are functions of C and the primes denote differentiation with respect to x. In all theorems dealing with the adjoint

$$y''' - (Py)'' + (Qy)' - Ry = 0$$

of (1.1) we make the additional assumption that $P(x)$ and $Q(x)$ are functions of C'' and C', respectively. Unless otherwise noted, the interval under consideration is $(0, \infty)$.

The oscillatory properties of equation (1.1) were first investigated in a classical paper by G. D. Birkhoff [1], which appeared in 1911. Further results were obtained in papers by Mammana [5] and Sansone [7]; the latter, which appeared in 1948, contains a complete bibliograpy. More recent work on this equation can be found in [2], [3], [8], and [9].

A solution of (1.1) will be called *oscillatory* if it has an infinity of zeros in $(0, \infty)$ and *nonoscillatory* if it has but a finite number of zeros in this interval. An equation is termed *oscillatory* if there exists at least one oscillatory solution, and *nonoscillatory* if all its solutions are nonoscillatory. This latter definition is necessary since an equation (1.1) may have both oscillatory and nonoscillatory solutions. Also, we say that (1.1) is nonoscillatory in (a, ∞) if none of its solutions has more than *two* zeros in (a, ∞). The number two is essential since there always exist solutions of (1.1) which have zeros at two arbitrary points.

In the study of the second-and fourth-order differential equations the self-adjoint forms are of special importance. The self-adjoint form of the third-order equation is

$$y''' + py' + \tfrac{1}{2}p'y = 0 \,. \tag{1.2}$$

The general solutions of (1.2) is $y = c_1u^2 + c_2uv + c_3v^2$, where $u(x)$ and $v(x)$ are linearly independent solutions of the second-order equation

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