# CONDITIONS FOR THE MODULARITY OF AN ORTHOMODULAR LATTICE

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**1. Introduction.** An *orthomodular lattice* is a lattice $L$ with 0 and 1 which is equipped with an *orthocomplementation* $' : L \to L$ and which satisfies the *orthomodular identity* $e \leqq f \Rightarrow f = e \vee (f \wedge e')$. Recall that an orthocomplementation $' : L \to L$ maps each element $e \in L$ onto a complement $e'$ of $e$ in $L$ in such a way that $e'' = e$ and $e \leqq f \Rightarrow f' \leqq e'$ for $e, f \in L$. The "logic" of (non-relativistic) quantum mechanics, i.e., the lattice of closed subspaces of a separable infinite dimensional Hilbert space [5, p. 49], as well as the "logic" of classical mechanics, i.e., the Boolean algebra of all Borel subsets of phase space modulo Borel subsets of measure zero [5, p. 48], are both instances of orthomodular lattices.

L. H. Loomis has shown in [4] that orthomodular lattices provide a natural environment for the abstract study of the dimension theory of operator algebras. I. Kaplansky [3] has obtained an elegant theorem to the effect that if an orthomodular lattice is complete and modular, then it is a continuous geometry.

An *involution semigroup* is a semigroup $S$ equipped with an *involution* $^*$, i.e., an antiautomorphism $^* : S \to S$ of period 2. An element $e \in S$ is called a *projection* in case $e = e^* = e^2$. In this paper, we use the term *Baer $^*$-semigroup* to refer to an involution semigroup $S$ (with a two-sided zero element 0) which is equipped with a mapping $' : S \to S$ such that

(i) $x'$ is a projection for $x \in S$ and

(ii) for $x \in S$, $\{y \mid y \in s \text{ and } xy = 0\} = x'S$. A projection $e \in S$ is said to be *closed* in case $e = e''$, and the collection of all closed projections in $S$ is denoted by $P' = P'(S)$. The notion of a Baer $^*$-semigroup was introduced in [2, § 2] in a slightly more general form.

In [2] it is shown that there is an intimate connection between orthomodular lattices and Baer $^*$-semigroups, namely: *If $S$ is a Baer $^*$-semigroup, then $P'(S)$ is an orthomodular lattice with $e \to e'$ as orthocomplementation and with partial order defined by $e \leqq f \Longleftrightarrow ef = e$ for $e$, $f \in P'(S)$. The element $0' = 1$ acts as a unit in the semigroup $S$. Conversely, every orthomodular lattice $L$ is isomorphic to a lattice $P'(S)$ for some Bear $^*$-semigroup $S$.*

In the sequel, the symbol $L$ always denotes an orthomodular lattice and the symbol $S$ always denotes a Baer $^*$-semigroup. When $S$ and $L$ are so related that there is an orthocomplementation preserving iso-

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