# WIRTINGER-TYPE INTEGRAL INEQUALITIES

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**1. Introduction.** The following inequalities (and other similar ones) are known:

(i) if $u'(x) \in L_2$ and $u(0) = 0$, then

$$\int_0^{\pi/2} u^2 dx \leqq \int_0^{\pi/2} u'^2 dx \qquad [4];$$

(ii) if $u''(x) \in L_2$ and $u(0) = u(\pi) = 0$, then

$$\int_0^{\pi} u^2 dx \leqq \int_0^{\pi} u''^2 dx \qquad [3];$$

in each case, equality occurs if and only if $u(x) \equiv A \sin x$. P. R. Beesack [1] has generalized these two types of inequalities by considering the underlying differential equations $y'' + py = 0$ and $y^{(iv)} - py = 0$ respectively, together with the equations satisfied by $y'/y$. In [2], a relation was obtained between the equation $y^{(2n)} - py = 0$ and the inequality

$$(-1)^n \int_a^b pu^2 dx \leqq \int_a^b u^{(n)^2} dx .$$

In this paper we let $Ly$ be the general self-adjoint linear operator of even order

$$\sum_{i=0}^{n} (f_i y^{(i)})^{(i)}$$

and extend the methods of [2] to relate the equation

$$Ly = 0 \tag{1}$$

and the inequalities

$$0 \leqq \sum_{i=0}^{n} (-1)^{n+i} \int_a^b f_i u^{(i)^2} dx \tag{2}$$

and

$$0 \geqq \int_a^b \frac{1}{f_n} \cdot u^2 dx + (-1)^n \int_a^b \frac{1}{f_0} \cdot u^{(n)^2} dx . \tag{3}$$

**2. Notation and lemmas.** Let $y_i = f_i y^{(i)}$, $v_i = \sum_{k=0}^{i} y_{n-k}^{(i-k)}$,

$$u_{ij} = v_{n-i}/y^{(j)}, \text{ and } y_{ij} = y^{(i)}/y^{(j)} \qquad (i = 0, \cdots, n) .$$

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