THE SECOND CONJUGATE SPACE OF A BANACH ALGEBRA AS AN ALGEBRA

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1. Introducion. A procedure has been given by Arens [1, 2] for defining a multiplication in the second conjugate space of a Banach algebra which makes that space into another Banach algebra. This idea was used with great effectiveness by Day [3] in his study of amenable semigroups.

We undertake here a rather systematic study of this notion. We begin in § 3 with a discussion of the second conjugate space $L^{**}(\mathfrak{G})$ of the group algebra $L(\mathfrak{G})$ of a locally compact group $\mathfrak{G}$ and its radical $\mathfrak{R}^{**}$. Suppose that $\mathfrak{G}$ is abelian and infinite. It is shown that $L^{**}(\mathfrak{G})$ is never semi-simple and never commutative; if $\mathfrak{G}$ is compact then $\mathfrak{R}^{**}$ is the annihilator in $L^{**}(\mathfrak{G})$ of that subset of the first conjugate space $L^{*}(\mathfrak{G})$ which can be identified with the continuous functions on $\mathfrak{G}$. For any locally compact abelian group $\mathfrak{G}$ let $\mathfrak{Y}$ be the subspace of $L^{*}(\mathfrak{G})$ that may be identified with the almost periodic functions on $\mathfrak{G}$, and let $\mathfrak{C}$ be the subspace of $L^{*}(\mathfrak{G})$ that may be identified with the continuous functions on $\mathfrak{G}$ vanishing at infinity. Let $\mathfrak{Y}^{\perp}$ and $\mathfrak{C}^{\perp}$ denote respectively the annihilators of $\mathfrak{Y}$ and $\mathfrak{C}$ in $L^{**}(\mathfrak{G})$. Then $L^{**}(\mathfrak{G})/\mathfrak{Y}^{\perp}$ is isometrically isomorphic as a Banach algebra to the measure algebra on the almost periodic compactification of $\mathfrak{G}$, and $L^{**}(\mathfrak{G})/\mathfrak{C}^{\perp}$ is isometrically isomorphic to the measure algebra on $\mathfrak{G}$. It is then abundently clear that the Arens multiplication in $L^{**}(\mathfrak{G})$ is intimately connected with much studied objects defined in terms of $\mathfrak{G}$.

In § 4 we observe a phenomenon which does not hold in the group algebra case. In the latter case we started with a commutative, semi-simple Banach algebra $B = L(\mathfrak{G})$ and obtained a second conjugate space B^{**} neither commutative nor semi-simple. Here we give of an example where B is commutative and semi-simple and B^{**} is not semi-simple but commutative.

We can consider B as embedded in B^{**} in the canonical way. In § 5 it is shown, for example, that each regular maximal (left, right or two-sided) and each primitive ideal is contained in an ideal of the same type in B^{**}. Also if B is commutative its radical is contained in the radical of B^{**}.

In § 6 it is shown that if T is a continuous homomorphism of B_1 into B_2, where B_k is a Banach algebra, then T^{**} is continuous homomorphism of B_1^{**} into B_2^{**} where these are considered as algebras.

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