LINEAR RECURRENCES OF ORDER TWO

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1. Introduction. A sequence of rational integers $\{f(n)\}$ satisfying a relation

$$f(n+k) = \sum_{i=1}^{k} A_i f(n+k-i)\,, \qquad A_k \neq 0\,,$$

where the A_i are rational integers, is called an *integral linear recurrence of order k*. Given such a linear recurrence and an integer c, one would like to know for what n does $f(n) = c$? In a very few particular instances (e.g. see [2], [6]) this question has been answered, but in general the question is very difficult. A less exacting problem is the determination of upper and lower bounds on the number, $M(c)$, of distinct n for which $f(n) = c$. We shall call $M(c)$ the multiplicity of c in the recurrence.

Much work has been done by C. L. Siegel [4], K. Mahler [3], Morgan Ward [9], [10], [11] and others concerning the multiplicity of 0 and the pattern of the appearance of 0 in the recurrence. Quite often from information on the multiplicity of 0 in one recurrence one can infer a bound on the multiplicity of all integers in another recurrence. However, as much of the information available concerning the zeros of a recurrence is for recurrences satisfying special conditions on the A_i, one cannot always ascertain in this way whether $M(c)$ is bounded.

Define the *multiplicity of a recurrence* as the least upper bound of the $M(c)$, as c ranges over the integers; and say that the multiplicity of the recurrence is *strictly infinite* if for some integer c, $M(c)$ is infinite. We are interested in examining the following questions:

(I) When is the multiplicity of a recurrence finite? When infinite?

(II) If the multiplicity of a recurrence is finite, what is it or at least what is an upper bound for it?

(III) Can the multiplicity of a recurrence be infinite and not strictly infinite?

Here, we confine our attention to recurrences of order 2. In the direction of the above questions, there is a conjecture that for a recurrence of order 2 either the multiplicity is strictly infinite or it is bounded above by 5. We are unable to resolve this conjecture, but we do obtain reasonably satisfactory answers to the questions for all recurrences of order 2 having $(A_1, A_2) = 1$.

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