# BEST FIT TO A RANDOM VARIABLE BY A RANDOM VARIABLE MEASURABLE WITH RESPECT TO A $\sigma$-LATTICE

H. D. BRUNK

**1. Introduction and summary.** Let $(\Omega, \mathscr{S}, \mu)$ be a probability space and $f$ a random variable, an $\mathscr{S}$-measurable function from $\Omega$ into the space $R$ of real numbers. Let $\mathscr{S}_0$ be a sub-$\sigma$-algebra of $\mathscr{S}$. Let $f$ be integrable; that is, let its expectation $E(f)$ exist. Then the Radon-Nikodym Theorem yields an $\mathscr{S}_0$-measurable function $g$, the conditional expectation of $f$ given $\mathscr{S}_0$: $g = E(f \mid \mathscr{S}_0)$. The conditional expectation $g$ is, in a strong sense to be made precise below, the best fit to $f$ by an $\mathscr{S}_0$-measurable function. The purpose of the present note is to show that there corresponds to $f$ a function with the same minimizing properties when an arbitrary sub-$\sigma$-lattice $\mathscr{L}$ takes the place of $\mathscr{S}_0$.

The conditional expectation $g = E(f \mid \mathscr{S}_0)$ has the property that

$$\int (f - g) h d\mu = 0$$

for $\mathscr{S}_0$-measurable $h$ such that the integral exists. It is then immediate that

$$\int (f - h)^2 d\mu = \int (f - g)^2 d\mu + \int (g - h)^2 d\mu .$$

More generally, the squared difference may be replaced by the W. H. Young form $\Delta_\Phi(\circ, \circ)$ determined by an arbitrary convex function $\Phi$ (see §2):

$$\int \Delta_\Phi(f, h) d\mu = \int \Delta_\Phi(f, g) d\mu + \int \Delta_\Phi(g, h) d\mu$$

for $\mathscr{S}_0$-measurable $h$, provided appropriate integrals exist. (The function $\Delta_\Phi(\circ, \circ)$ is nonnegative and vanishes when the arguments are equal.) Thus, for every $\Phi$, $g = E(f \mid \mathscr{S}_0)$ is the solution of the minimizing problem: given $f$, to minimize $\int \Delta_\Phi(f, h) d\mu$ in the class of $\mathscr{S}_0$-measurable functions. The conditional expectation therefore enjoys a powerful claim to be the "best" fit to $f$ by an $\mathscr{S}_0$-measurable function. (Blackwell [3] has remarked that for square-integrable functions, the conditional expectation may be regarded as a projection in Hilbert space.)

Received June 15, 1960. Part of the research for this paper was done while the author was a recipient of a U. S. Government Award under Public Law 584 of the 79th Congress—The Fulbright Act, and part was supported by the U. S. Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command under Contract AF 18(600)-1108.