A GENERALIZATION OF THE STONE-WEIERSTRASS THEOREM

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1. Introduction. Consider a compact Hausdorff space X and the set $C(X)$ of all continuous complex-valued functions on X. Consider also a subset $\mathfrak{A}$ of $C(X)$ which is an algebra, which is closed in the uniform topology of $C(X)$, which contains the constant functions, and which contains sufficiently many functions to distinguish points of X. Such an algebra $\mathfrak{A}$ is called *self-adjoint* if the complex conjugate of each function in $\mathfrak{A}$ is in $\mathfrak{A}$. The classical Stone-Weierstrass Theorem states that if $\mathfrak{A}$ is self-adjoint then $\mathfrak{A} = C(X)$. If $\mathfrak{A}$ has the property that the only functions in $\mathfrak{A}$ which are real at every point of X are the constant functions then $\mathfrak{A}$ is called *anti-symmetric*. Clearly anti-symmetry and self-adjointness are opposite properties, in the sense that if $\mathfrak{A}$ has both properties then X must consist of a single point.

Hoffman and Singer [2] have studied these two properties and given several interesting examples. The present paper was inspired by their work but it more directly relates to a previous paper of Šilov [3]. The purpose of the present paper is to prove the following decomposition theorem for a general algebra $\mathfrak{A}$ of the type defined above.

THEOREM. *There exists a partition P of X into disjoint closed sets such that*

(i) *for each S in P the restriction $\mathfrak{A}_S$ of $\mathfrak{A}$ to S is anti-symmetric,*

(ii) *if a function f in $C(X)$ has, for each S in P, a restriction to S which belongs to $\mathfrak{A}_S$, then f is in $\mathfrak{A}$,*

(iii) *for each S in P, each closed subset T of $X - S$, and each $\varepsilon > 0$ there exists g in $\mathfrak{A}$ with $\|g\| \leqq 1$, with $|g(x) - 1| < \varepsilon$ for x in S, and with $|g(x)| < \varepsilon$ for x in T.*

Property (ii) of this theorem is the essential new fact of this paper. The construction given below which leads to the partition P is due to Šilov [3], who in essence proved (i) and (iii). Šilov proved a weaker property than (ii). Our proofs are different from those of Šilov, although the construction is the same.

The fact that the Stone-Weierstrass theorem is a special case of the theorem to be proved here is clear. If $\mathfrak{A}$ is self-adjoint then each $\mathfrak{A}_S$ is self-adjoint. Since $\mathfrak{A}_S$ is also anti-symmetric, each set S in P consists of a single point. Therefore $\mathfrak{A}_S = C(S)$. By the theorem to

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