ASYMPTOTIC ESTIMATES FOR LIMIT CIRCLE PROBLEMS

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1. Preliminaries. Characteristic value problems will be considered for the second order, ordinary, linear differential operator L defined by

$$Lx = \frac{1}{k(s)}\left\{-\frac{d}{ds}\left[p(s)\frac{dx}{ds}\right] + q(s)x\right\} \tag{1.1}$$

on the open interval $\omega_- < s < \omega_+$, where k, p, q are real-valued functions on this interval with the properties that

(i) p is differentiable;

(ii) k and q are piecewise continuous; and

(iii) k and p are positive-valued. The points ω_- and ω_+ are in general singularities of L; the possibility that they are $\pm\infty$ is not excluded. It will be convenient to use the notations

$$(x, y)_s^t = \int_s^t x(u)\bar{y}(u)k(u)du, \qquad \omega_- \leqq s < t \leqq \omega_+ , \tag{1.2}$$

$$[xy](s) = p(s)[x(s)\bar{y}'(s) - x'(s)\bar{y}(s)] . \tag{1.3}$$

Then Green's symmetric formula for L has the form

$$(Lx, y)_s^t - (x, Ly)_s^t = [xy](t) - [xy](s) . \tag{1.4}$$

The symbols $[xy](\pm)$ will be used as abbreviations for the limits of $[xy](s)$ as $s \to \omega_\pm$, and (x, y) will be used for the left member of (1.2) when s, t have been replaced by ω_-, ω_+. Let $\mathfrak{H}$, $\mathfrak{H}_{ab}$ denote the Hilbert spaces which are the Lebesgue spaces with respective inner products (x, y), $(x, y)_a^b$ and norms $\|x\| = (x, x)^{1/2}$, $\|x\|_a^b = [(x, x)_a^b]^{1/2}$, $\omega_- \leqq a < b \leqq \omega_+$.

Let a_0 and b_0 be fixed numbers satisfying $\omega_- < a_0 < b_0 < \omega_+$ and let R_0 be the rectangle in the $a-b$-plane described by the inequalities $\omega_- < a \leqq a_0$, $b_0 \leqq b < \omega_+$. Every closed, bounded subinterval $[a, b]$ of the basic interval (ω_-, ω_+) can be associated in a one-to-one manner with a point in R_0. For every such $[a, b]$ we shall consider the regular Sturm-Liouville problem

$$Ly = \mu y, \qquad U_a y = U_b y = 0 \tag{1.5}$$

on $[a, b]$, where U_a, U_b are the linear boundary operators

$$\begin{aligned} U_a y &= \alpha_0(a)y(a) + \alpha_1(a)y'(a) \\ U_b y &= \beta_0(b)y(b) + \beta_1(b)y'(b) , \end{aligned} \tag{1.6}$$

with α_0, α_1 real-valued functions not both 0 for any value of a on $(\omega_-, a_0]$,

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