# CENTROID SURFACES

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**1. Introduction.** Let $M_1, \cdots, M_{n-1}$ denote $(n-1)$ bounded closed sets in $E_n$. Busemann [1] has established the expression

$$(1.1)\quad |M_1| \cdots |M_{n-1}| = \frac{(n-1)!}{2} \int_{\Omega_n} \left( \int_{M_1(u)} \cdots \int_{M_{n-1}(u)} T(z, p_1, \cdots, p_{n-1}) dV_{p_1}^{n-1} \cdots dV_{p_{n-1}}^{n-1} \right) d\omega_u^n$$

where $|M_i|$ is the $n$-dimensional Lebesgue measure or volume of $M_i$. On the righthand side $M_i(u)$ is the cross-section of $M_i$ with the hyperplane through $z$ normal to the unit vector $u$, the point $p_i$ varies in $M_i(u)$ and the differential $dV_{p_i}^{n-1}$ is the $(n-1)$-dimensional volume element of $M_i(u)$ at $p_i$. The final integration is extended over the surface $\Omega_n$ of the solid-unit sphere $U_n$ and $d\omega_u^n$ is the area element of $\Omega_n$ at point $u$. By $T(z, p_1, \cdots, p_r)$ we will denote the $r$-dimensional volume of the simplex (possibly degenerate) with vertices $z, p_1, \cdots, p_r$.

Let

$$(1.2)\qquad \pi_r = \frac{\pi^{r/2}}{\Gamma(r/2+1)}\,.$$

For $n \geqq 3$, Busemann also shows by Steiner's symmetrization that

$$(1.3)\quad |M_1| \cdots |M_{n-1}| \geqq \frac{1}{n} \frac{\pi_n^{n-2}}{\pi_{n-1}^n} \int_{\Omega_n} |M_1(u)|^{n/(n-1)} \cdots |M_{n-1}(u)|^{n/(n-1)} d\omega_u^n$$

for nondegenerate convex bodies $M_i$ where the equality sign holds only when the $M_i$ are homothetic solid ellipsoids with center $z$. Here $|M_i(u)|$, of course, denotes the $(n-1)$-dimensional volume of $M_i(u)$. In this regard we will also, as a matter of convenience, not index lower dimensional mixed discriminates and mixed volumes since the dimension will be evident from the number of components.

The primary purpose of this note is to reinterpret (1.1) as an integration of the type (1.3) retaining the equality sign. This is given in § 3 by (3.20). In addition other integral expressions and inequalities are derived which are geometrically of the same type as those considered above.

**2. Fenchel's momental ellipsoid.** Let $M$ be a bounded closed set with positive volume. The centroid $s$ of $M$ is defined by its rectangular coordinates

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