# A THEOREM ON REGULAR MATRICES

PETER PERKINS

In this paper it will be proved that if any nonnegative, square matrix $P$ of order $r$ is such that $P^m > 0$ for some positive integer $m$, then $P^{r^2-2r+2} > 0$. This result has already appeared in the literature, [2], but the following is a complete and elementary proof given in detail except for one theorem of I. Schur in [1] which is stated without proof. The term regular is taken from Markov chain theory[1] in which a regular chain is one whose transition matrix has the above property.

A *graph* $G_P$ associated with any nonnegative, square matrix $P$ of order $r$ is a collection of $r$ distinct points $S = \{s_1, s_2, \cdots, s_r\}$, some or all of which are connected by directed lines. There is a directed line (indicated pictorially by an arrow) from $s_i$ to $s_j$ in the graph $G_P$ if and only if $p_{ij} > 0$ in the matrix $P = (p_{ij})$. A *path sequence* or *path* in $G_P$ is any finite sequence of points of $S$ (not necessarily distinct) such that there is a directed line in $G_P$ from every point in the sequence to its immediate successor. The *length* of a path is one less than the number of occurrences of points in its sequence. A *cycle* is any path that begins and ends with the same point and a *simple cycle* is a cycle in which no point occurs twice except, of course, for the first (and last). Two cycles are *distinct* if their sequences are not cyclic permutations of each other. A nonnegative, square matrix $P$ is *regular* if $P^m > 0$ for some positive integer $m$. Likewise, a graph $G_P$ associated with a nonnegative, square matrix $P$ is *regular* if there exists a positive integer $m$ such that an infinite set of paths $A_0, A_1, \cdots, A_n, \cdots$ can be found, the length of each path being $L_n = m + n$, $n = 0, 1, 2, \cdots$. The usual notation $p_{ij}^{(m)}$ is used to denote the $ij$th entry of the matrix $P^m$. In all that follows we shall consider only regular matrices $P$ and their associated graphs $G_P$.

Some immediate consequences of these definitions and the definition of matrix multiplication are the following:

(1) There is a path $s_{k_1} \cdots s_{k_{m+1}}$ in $G_P$ if and only if $p_{k_1 k_{m+1}}^{(m)} > 0$ in $P^m$.
(2) $P$ is regular if and only if $G_P$ is regular.
(3) There exists some path from any point in $G_P$ to any point in $G_P$.
(4) For any given $i$ and $j$ there exists some $m$ such that $p_{ij}^{(m)} > 0$.
(5) If $P^m > 0$ then $P^{m+n} > 0$, $n = 0, 1, 2, \cdots$.

Let $C = \{C_1, C_2, \cdots, C_t\}$ be all the distinct simple cycles of $G_P$ and $\{c_1, c_2, \cdots, c_t\}$ be the corresponding lengths.

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[1] This is as treated by Kemeny and Snell in [3].