GAME THEORETIC PROOF THAT CHEBYSHEV INEQUALITIES ARE SHARP

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1. Summary. This paper is concerned with showing that Chebyshev inequalities obtained by the standard method are sharp. The proof is based on relating the bound to the solution of a game. An optimum strategy yields a portion of the extremal distribution, and the remainder is obtained as a solution of the relevant moment problem.

2. Introduction. Let X be a random vector taking values in $\mathscr{X} \subset R^k$, and suppose that $Ef(X) \equiv E(f_1(X), \cdots, f_r(X)) = (\varphi_1, \cdots, \varphi_r) \equiv \varphi$ is given, where f_j is a real valued function on $\mathscr{X}$. For convenience, we suppose $f_1 \equiv 1$. An upper bound for $P\{X \in \mathscr{T}\}$, $\mathscr{T} \subset \mathscr{X}$, may be obtained as follows. If $a = (a_1, \cdots, a_r) \in R^r$ and $\chi_{\mathscr{T}}$ is the indicator of $\mathscr{T}$ then $af' \geqq \chi_{\mathscr{T}}$ on $\mathscr{X}$ implies $P\{X \in \mathscr{T}\} \leqq a\varphi'$, and if $\mathscr{A}_0 = \{a: af' \geqq \chi_{\mathscr{T}}$ on $\mathscr{X}\}$, a "best" bound is given by

$$P\{X \in \mathscr{T}\} \leqq \inf_{a \in \mathscr{A}_0} a\varphi' . \tag{2.1}$$

In general, a bound is called sharp if it cannot be improved. For some cases, when $\mathscr{T}$ is assumed to be closed, the bound can actually be attained by a distribution satisfying the moment hypotheses.

The main result of this paper is

THEOREM 2.1. *Inequality* (2.1) *is sharp in the following cases.*

(I) $X = (X_1, \cdots, X_k)$ with EX_iX_j or EX_i and EX_iX_j given, $i, j = 1, \cdots, k$.

(II) X has range $(-\infty, \infty), [0, \infty)$, or $[0, 1]$, and EX^j is given, $j = 1, \cdots, m$.

(III) X is a random angle in $[0, 2\pi)$ and the trigonometric moments $Ee^{i\alpha X}, \alpha = \pm 1, \cdots, \pm m$ are given.

Sharpness has been shown in (I) by Marshall and Olkin [6] when $\mathscr{T}$ is convex, and by Isii [3, 4] in the unbounded cases of (II). Sharpness has also been proved in a number of specialized situations.

In § 3 the proof for (I) will be given in detail. The necessary alterations for each of the remaining cases will be given in § 4, 5, 6, 7. The solution of certain moment problems depend on conditions on Hankel matrices, i.e., matrices of the form $H = (h_{i+j})$, and some results concerning these matrices are given in § 8.

Received August 31, 1960. Research sponsored by the Office of Naval Research at Stanford University.