THE SPECTRUM OF SINGULAR SELF-ADJOINT ELLIPTIC OPERATORS

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This note deals with the Dirichlet problem for the second order elliptic operator

$$L = -\frac{1}{r(x)} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial}{\partial x_i}\right) + c(x)$$

whose coefficients are defined in a bounded domain $G \subset E^n$. We suppose the following:

(a) The $a_{ij}(x)$ are complex valued and of class C' in G; $a_{ij} = \bar{a}_{ji}$.
(b) $c(x)$ is real valued, continuous, and bounded below in G.
(c) $r(x)$ is continuous and positive in G.
(d) There exists a function $\sigma(x)$, continuous and positive in G satisfying

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\bar{\xi}_j \geqq \sigma \sum_{i=1}^{n} |\xi_i|^2$$

for all x in G and all complex n-tuples $\vec{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$.

Under these assumptions it is easy to show that L is formally self-adjoint in the Hilbert space $\mathscr{L}^2_r(G)$ of functions which satisfy

$$\int_G r\,|u|^2\,dx < \infty\,.$$

We denote by $C_0^\infty(G)$ the set of infinitely differentiable functions with compact support in G. The operator L defined on $C_0^\infty(G)$ is a semi-bounded symmetric operator in $\mathscr{L}^2_r(G)$ and therefore has a Friedrichs extension which is self-adjoint in $\mathscr{L}^2_r(G)$. This operator, to be denoted by $\bar{L}$, will be referred to as the Dirichlet operator associated with L on G. It is well known that $\bar{L}$ is unique, has the same lower bound as the symmetric operator L, and that in sufficiently regular cases, $\bar{L}$ can be obtained by imposing Dirichlet boundary conditions on the domain of L^*. The purpose of this note is to state a criterion for the discreteness of the spectrum of $\bar{L}$.

We shall say that the spectrum of an operator A is discrete if the spectrum of A consists of a set of isolated eigenvalues of finite multiplicity. The complex number λ belongs to the essential spectrum of A if there exists an orthonormal sequence $\{u_n\}$ it the domain of A for which $(A - \lambda I)u_n \to 0$. If A is self-adjoint, then it can be shown (see

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