QUOTIENT RINGS OF RINGS WITH ZERO SINGULAR IDEAL

R. E. JOHNSON

Many papers have been written recently (see [2]-[14] of bibliography) on extensions of rings to rings of quotients. In most of these papers, strong enough conditions are imposed on the given rings to insure that each has a vanishing singular ideal (first defined in [5]). It seems appropriate at this time to collect these results and present them in as general a form as possible. In this paper, it is assumed that each ring has a zero right singular ideal. A subsequent paper will give the quotient structure of a ring having a vanishing right and left singular ideal.

1. Introduction. If R is a ring and M is an R-module, then $L(R)$ and $L(M, R)$ will designate the lattices of right ideal of R and R-submodules of M, respectively. Superscripts "r" and "l" will be used in designating the right and left annihilators, respectively, of an element or subset of a ring or module. The context will always make it clear from what set the annihilators are to be chosen.

In a lattice L with 0 and I, an element B is called an *essential extension* of element A, and we write $A \subset' B$, if and only if $A \subset B$ and $C \cap A \neq 0$ for every C in L for which $C \cap B \neq 0$. An element A of L is called *large* if $A \subset' I$. The sublattice of L of all large elements is designated by $L^{\blacktriangle}$.

If R is a ring and M is a right R-module, then let

$$M^{\blacktriangle}(R) = \{x \mid x \in M, x^r \in L^{\blacktriangle}(R)\}, \quad R^{\blacktriangle} = \{x \mid x \in R, x^r \in L^{\blacktriangle}(R)\}.$$

It is easily shown that $M^{\blacktriangle}(R)$ is a submodule of M and $R^{\blacktriangle}$ is a (two-sided) ideal of R. The ideal $R^{\blacktriangle}$ is called the *singular ideal* [5; p. 894] of R.

A ring R with zero singular ideal has the unusual property, proved in [7; Section 6], that each $A \in L(R)$ has a unique maximal essential extension A^s in $L(R)$. The mapping $s: A \to A^s$ of $L(R)$ is shown there to be a closure operation on $L(R)$ having the following properties:

(1) $0^s = 0$,

(2) $(A \cap B)^s = A^s \cap B^s$ for each $A, B \in L(R)$, and

(3) $(x^{-1}A)^s = x^{-1}A^s$ for each $x \in R$ and $A \in L(R)$, where $x^{-1}B = \{y \mid y \in R, xy \in B\}$. The set $L^s(R)$ of closed right ideals (i.e., $A = A^s$) may be made into a lattice in the usual way by defining the union of a set of

Received December 10, 1960. This research was supported in part by a grant from the National Science Foundation.