RELATIVE SELF-ADJOINT OPERATORS IN HILBERT SPACE

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1. Introduction. Let A be a closed operator from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{H}'$. The main purpose of this present paper is to develop a spectral theory for an operator A of this type. This theory is analogous to the given in the self-adjoint case and reduces to the standard theory when A is self-adjoint. The spectral theory here given is based on generalization of the concept of self-adjointness. Let A^* denote the adjoint of A. An operator T on $\mathfrak{H}$ to $\mathfrak{H}'$ will be said to be an elementary operator if $TT^*T = T$. If T is elementary, the operator TA^*T can be considered to be an adjoint of A relative to T. If $A = TA^*T$, then A will be said to be self-adjoint relative to T. The polar decomposition theorem for A implies the existence of a unique elementary operator R relative to which A is self-adjoint and having the further property that R has the same null space as A and that A^*R is a nonnegative self-adjoint operator in the usual sense. Every elementary operator T relative to which A is self-adjoint is of the form $T = T_0 + R_1 - R_2$, where $R = R_1 + R_2$ and T_0, R_1, R_2 are $*$-orthogonal. Two operators B and C are said to be $*$-orthogonal if $B^*C = 0$ and $BC^* = 0$ on dense sets in $\mathfrak{H}$ and $\mathfrak{H}'$ respectively.

An operator B will be called a section of an operator A if there is an operator C $*$-orthogonal to B such that $A = B + C$. If R is the elementary operator associated with A, there exists a one parameter family A_λ, R_λ $(0 < \lambda < \infty)$ of sections of A, R respectively such that R_λ is the elementary operator belonging to A_λ, $\| A_\lambda \| \leqq \lambda$, A_μ $(\mu < \lambda)$ is a section of A_λ and $A = \int_0^\infty \lambda dR_\lambda$. From this result it is seen that A possesses a spectral decomposition relative to any elementary operator T relative to which A is self-adjoint. These results can be extended to the case in which A is normal relative to T. When $\mathfrak{H}' = \mathfrak{H}$ and T is the identity, these results give the usual spectral theory for self-adjoint operators. Examples are given in §§ 4 and 10 below. In particular spectral resolutions are given for the gradient of a function and its adjoint, the divergence of a vector. The finite dimensional case has been treated in a recent paper by the author[1].

The results given below are elementary in nature and are based

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[1] M. R. Hestenes, Relatively hermitian matrices, to be published in the Pacific Journal of Mathematics.