BASES OF TENSOR PRODUCTS OF BANACH SPACES

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1. Introduction. In this note we use the conventions and notations of Schatten [4] with the exception that we use B' to indicate the dual (conjugate) space of a Banach space B and $\langle x, x' \rangle$ as the action of an element x and a functional x' on each other. Schatten defines the tensor product $B_1 \otimes {}_\alpha B_2$ as the completion of the algebraic tensor product $B_1 \otimes B_2$ of two Banach spaces B_1 and B_2, on which the cross norm α has been imposed. We discuss the proposition, "If B_1 and B_2 have Schauder bases, then $B_1 \otimes {}_\alpha B_2$ has a Schauder basis." We prove this for $\alpha = \gamma$ ($B_1 \otimes {}_\gamma B_2$ is the trace class of transformations of B_1' into B_2). We also prove it for $\alpha = \lambda$ ($B_1 \otimes {}_\lambda B_2$ is the class of all completely continuous linear transformations of B_1' into B_2) in the case in which the bases of B_1 and B_2 satisfy an "isometry condition". This condition is not very restrictive. We know of no instance in which it is not satisfied. Next we show that unconditional bases of B_1 and B_2 do not necessarily yield an unconditional basis for the tensor product, even in the nicest conceivable infinite dimensional case, that in which $B_1 = B_2 =$ Hilbert space, and the bases are orthonormal and identical.

We recall certain facts about Schauder bases, and set some general notation that we use throughout the paper. We usually work with a biorthogonal set $\Omega = \{x_i, x_i'\}_i$ associated with a Banach space B, so that $\chi = \{x_i\}_i$ is a basis for B with coefficients supplied by the corresponding sequence of functionals $\chi' = \{x_i'\}_i$. We will have to do with the closed linear manifold B^Ω of B' generated by the elements of χ'. Since B and B^Ω are in duality it is possible to embed B in $(B^\Omega)'$ by the same formula that effects the embedding of B in B''. We denote by ${}_nP_m$ the projection of B defined by ${}_nP_m x = \sum_{i=n}^m \langle x, x_i' \rangle x_i$. The double sequence $\{{}_nP_m\}_{n,m}$ is uniformly bounded. We denote by T' the transpose of any transformation T. The following lemma, given without proof, is but a trivial strengthening of [2, p. 18, Theorem 1].

LEMMA 1. *Let E be a dense vector subspace of B, Ω a biorthogonal set of B such that $\chi \subset E$, the vector space spanned by χ is dense in E and the sequence $\{{}_nP_m\}_{n,m}$ is uniformly bounded on E. Then Ω defines a basis for B.*

2. The tensor product of two biorthogonal sets. Let $\Omega_1 = \{x_i, x_i'\}_i$ be a biorthogonal set of B_1 and $\Omega_2 = \{y_i, y_i'\}_i$ a biorthogonal set of B_2.

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