THE WAVE EQUATION FOR DIFFERENTIAL FORMS

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1. The Problem. Let M be a compact C^∞ Riemannian manifold of dimension N, having a positive definite metric. The operator $\Delta = d\delta + \delta d$ (see [13] for notation) maps p-forms $(0 \leqq p \leqq N)$ into p-forms and it reduces, when $p = 0$, to minus the Laplace-Beltrami operator. Let $c(P)$ be a C^∞ function which is nonpositive for $P \in M$, and consider the Cauchy problem of solving the system

$$(1.1) \qquad \left(L + \frac{\partial^2}{\partial t^2}\right)v \equiv \left(\Delta + c + \frac{\partial^2}{\partial t^2}\right)v = f(P, t)$$

$$(1.2) \qquad v(P, 0) = g(P)\,, \qquad \frac{\partial}{\partial t} v(P, 0) = h(P)\,,$$

where f, g, h are C^∞ forms of degree p. The main purpose of the present paper is to solve the system (1.1), (1.2) by the method of Fourier.

The Cauchy problem for second order self-adjoint hyperbolic equations was solved by Fourier's method by Ladyzhenskaya [8] and more recently (with some improvements) by V. A. Il'in [6]. In [8], other methods are also described, namely: finite differences, Laplace transforms, and analytic approximations using a priori inequalities. Higher order hyperbolic equations were treated by Petrowski [12], Leray [9] and Garding [5].

The Fourier method can be based on the fact that the series

$$(1.3) \qquad \sum_{\lambda_n > 0} \frac{|\varphi_n(x)|^2}{\lambda_n^\alpha}, \ \sum_{\lambda_n > 0} \frac{|\partial\varphi_n(x)/\partial x|^2}{\lambda_n^{\alpha+1}}, \ \sum_{\lambda_n > 0} \frac{|\partial^2\varphi_n(x)/\partial x^2|^2}{\lambda_n^{\alpha+2}}$$

are uniformly convergent. Here $\{\varphi_n\}$ and $\{\lambda_n\}$ are the sequences of eigenfunctions and eigenvalues of the elliptic operator appearing in the hyperbolic equation. In [6] the convergence of (1.3) is proved for $\alpha = [N/2] + 1$. Our proof of the analogous result for eigenforms is different from that of [6] and yields a better (and sharp) value for α, namely, $\alpha = N/2 + \varepsilon$ for any $\varepsilon > 0$. It is based on asymptotic formulas which we derive for $\sum_{\lambda_n \leqq \lambda} |\partial^j\varphi_n(x)/\partial x^j|^2$ as $\lambda \to \infty$.

In § 2 we recall various definitions and introduce the fundamental solution for $L + \partial/\partial t$ which was constructed by Gaffney [4] in the case $c(P) \equiv 0$. In § 3 we derive some properties of the fundamental solution. These properties are used in § 4 to derive the asymptotic formulas for $\sum_{\lambda_n \leqq \lambda} |\partial^j\varphi_n(x)/\partial x^j|^2$, by which the convergence of the series in (1.3) for any $\alpha > N/2$ follows. In § 5 we solve the problem (1.1), (1.2); first for f, g, h

Received January 13, 1961. Prepared under Contract Nonr 710 (16) (NR 044 004) between the Office of Naval Research and the University of Minnesota.