MEAN CROSS-SECTION MEASURES OF HARMONIC MEANS OF CONVEX BODIES

WILLIAM J. FIREY

1. In [2] the notion of p-dot means of two convex bodies in Euclidean n-space was introduced and certain properties of these means investigated. For $p = 1$, the mean is more appropriately called the harmonic mean; here we restrict the discussion to this case. The harmonic mean of two convex bodies K_0 and K_1, which will always be assumed to share a common interior point Q, is defined as follows. Let $\hat{K}$ denote the polar reciprocal of K with respect to the unit sphere E centred at Q; let $(1-\vartheta)\hat{K}_0 + \vartheta\hat{K}_1$, with $0 \leqq \vartheta \leqq 1$, be the usual arithmetic or Minkowski mean of $\hat{K}_0$ and $\hat{K}_1$. The harmonic mean of K_0, K_1 is the convex body $[(1-\vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^\wedge$. In more analytic terms, if $F_i(x)$ are the distance functions with respect to Q of K_i, for $i = 0, 1$, then the body whose distance function with respect to Q is $(1-\vartheta)F_0(x) + \vartheta F_1(x)$ is the harmonic mean of K_0 and K_1.

In the paper mentioned, a dual Brunn-Minkowski theorem was established, namely

$$(1) \qquad V^{1/n}([(1-\vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^\wedge) \leqq 1\Big/\left[\frac{(1-\vartheta)}{V^{1/n}(K_0)} + \frac{\vartheta}{V^{1/n}(K_1)}\right]$$

where $V(K)$ means the volume of K. There is equality if and only if K_0 and K_1 are homothetic with the centre of magnification at Q.

Here we develop a more inclusive theorem regarding the behaviour of each mean cross-section measure, ("Quermassintegral") $W_\nu(K)$, $\nu = 0, 1, \cdots, n-1$, cf. [1]. The result is

$$(2) \qquad W_\nu^{1/(n-\nu)}([(1-\vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^\wedge) \leqq 1\Big/\left[\frac{(1-\vartheta)}{W_\nu^{1/(n-\nu)}(K_0)} + \frac{\vartheta}{W_\nu^{1/(n-\nu)}(K_1)}\right].$$

The cases of equality are just those of the dual Brunn-Minkowski theorem, $(\nu = 0)$.

2. We first list some preliminary items used in the proof of (2). We shall use Minkowski's inequality in the form

$$(3) \qquad \int[(1-\vartheta)f_0^p + \vartheta f_1^p]^{1/p}dx \leqq \left[(1-\vartheta)\left(\int f_0 dx\right)^p + \vartheta\left(\int f_1 dx\right)^p\right]^{1/p}.$$

Here the functions f_i are assumed to be positive and continuous over the closed and bounded domain of integration common to all the integrals,

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