EXTENSIONS OF HOMOMORPHISMS

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1. Introduction. A multiplication was introduced by R. Arens [1] [2] into the second conjugate space B^{**} of a Banach algebra, B, which made B^{**} into a Banach algebra. The algebra of the second conjugate space was studied by Civin and Yood [3], with particular attention given to the case where B was $L(\mathfrak{G})$, the group algebra of the locally compact abelian group $\mathfrak{G}$. Among the results they noted was that the algebra $M(\mathfrak{G})$ of finite regular Borel measures on $\mathfrak{G}$ was isomorphic as an algebra with a quotient algebra of $L^{**}(\mathfrak{G})$. With $\mathfrak{H}$ also a locally compact abelian group, P. J. Cohen showed [4, p. 220] that any homomorphism of $L(\mathfrak{G})$ into $M(\mathfrak{H})$ has an extension which was a homomorphism of $M(\mathfrak{G})$ into $M(\mathfrak{H})$.

In §3 we discuss the extensions of homomorphisms defined on a Banach algebra A into either the second conjugate algebra B^{**} of a Banach algebra B or certain of its quotient algebras. The result of Cohen quoted above is included in Theorem 3.7 when $\mathfrak{G}$ and $\mathfrak{H}$ are compact groups. In §4 we indicate, for compact $\mathfrak{H}$, a class of homomorphisms from $L(\mathfrak{G})$ into $M(\mathfrak{H})$, which are induced by homomorphisms of $L(\mathfrak{G})$ into $L^{**}(\mathfrak{H})$.

2. Notation. The notation of Civin and Yood [3] is used throughout. If A is a Banach algebra, A^*, A^{**}, $\cdots$ denote the various conjugate spaces of A. For $f \in A^*$, $x \in A$, $\langle f, x\rangle \in A^*$ is defined by $\langle f, x\rangle(y) = f(xy)$, $y \in A$. For $F \in A^{**}$, $f \in A^*$, $[F, f] \in A^*$ is defined by $[F,f](x) = F(\langle f, x\rangle)$, $x \in A$. Also for $F \in A^{**}$, $G \in A^{**}$ the multiplication FG is defined in A^{**} by $FG(f) = F([G, f])$, $f \in A^*$.

For some purposes, Arens [2] also considers a second multiplication $F\cdot G$ defined for F and G in A^{**} in a manner similar to the above, except that at the first stage, $\langle f | x\rangle \in A^*$ is defined by $\langle f | x\rangle(y) = f(yx)$, $f \in A^*$, $x, y \in A$. Arens calls the multiplication in A *regular* provided that $F\cdot G = GF$ for all $F, G \in A^{**}$. Clearly, if A is commutative, then A^{**} is commutative if and only if the multiplication in A is regular. The same notation as above, in terms of bilinear functionals, is used in the sequel with respect to a multiplication in A^{****} which comes from the first of the above multiplications in A^{**}.

If π is the natural mapping of A into A^{**}, we say that a mapping φ defined on A^{**} into a set $\mathfrak{S}$ is an extension of a mapping ρ defined on A into $\mathfrak{S}$ if $\varphi(\pi x) = \rho(x)$ for $x \in A$.

For any subset $\mathfrak{J}$ in A^*, we use the notation $\mathfrak{J}^{\perp}$ for $\{F \in A^{**} | F(f) = 0, f \in \mathfrak{J}\}$.

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