SOME CONGRUENCES FOR THE BELL POLYNOMIMALS

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1. Let $\alpha_1, \alpha_2, \alpha_3, \cdots$ denote indeterminates. The Bell polynomial $\phi_n(\alpha_1, \alpha_2, \alpha_3, \cdots)$ may be defined by $\phi_0 = 1$ and

$$\phi_n = \phi_n(\alpha_1, \alpha_2, \alpha_3, \cdots) = \sum \frac{n!}{k_1!(1!)^{k_1}k_2!(2!)^{k_2}\cdots}\alpha_1^{k_1}\alpha_2^{k_2}\cdots , \tag{1.1}$$

where the summation is over all nonnegative integers k_j such that

$$k_1 + 2k_2 + 3k_3 + \cdots = n .$$

For references see Bell [2] and Riordan [5, p. 36]. The general coefficient

$$A_n(k_1, k_2, k_3, \cdots) = \frac{n!}{k_1!(1!)^{k_1}k_2!(2!)^{k_2}\cdots} \tag{1.2}$$

is integral; this is evident from the representation

$$A_n(k_1, k_2, k_3, \cdots) = \frac{n!}{k_1!(2k_2)!(3k_3)!\cdots} \cdot \frac{(2k_2)!}{k_2!(2!)^{k_2}} \frac{(3k_3)!}{k_3!(3!)^{k_3}} \cdots$$

and the fact that the quotient

$$\frac{(rk)!}{k!(r!)^k}$$

is integral [1, p. 57].

The coefficient $A_n(k_1, k_2, k_3, \cdots)$ resembles the multinomial coefficient

$$M(k_1, k_2, k_3 \cdots) = \frac{(k_1 + k_2 + k_3 + \cdots)!}{k_1!k_2!k_3\cdots} .$$

If p is a fixed prime it is known [3] that $M(k_1, k_2, k_3, \cdots)$ is prime to p if and only if

$$k_i = \sum_j a_{ij}p^j \qquad (0 \leqq a_{ij} < p) ,$$

$$k_1 + k_2 + k_3 + \cdots = \sum_j a_j p^j \qquad (0 \leqq a_j < p)$$

and

$$\sum_i a_{ij} = a_j \qquad (j = 0, 1, 2, \cdots) .$$

It does not seem easy to find an analogous result for $A_n(k_1, k_2, k_3, \cdots)$. For some special results see § 3 below.

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