UPPER BOUNDS FOR THE EIGENVALUES OF SOME VIBRATING SYSTEMS

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1. Introduction. Let $p(x) \geqq 0$, $x \in [0, a]$, be the density of a string fixed at the points $x = 0$ and $x = a$ under unit tension. The natural frequencies of the string are determined by the eigenvalues of the differential system

$$u'' + \lambda p(x)u = 0,\ u(0) = u(a) = 0 . \tag{1}$$

We note that these eigenvalues depend on the density function $p(x)$ and denote them accordingly by

$$0 < \lambda_1(p) < \lambda_2(p) < \lambda_3(p) < \cdots .$$

M. G. Krein [5] has found the sharp bounds

$$\frac{4Hn^2}{M^2} X\left(\frac{M}{aH}\right) \leqq \lambda_n(p) \leqq \frac{\pi^2 n^2 H}{M^2} \qquad (n = 1, 2, \cdots)$$

where $X(t)$ is the least positive root of the equation

$$\sqrt{X} \tan X = \frac{t}{1 - t}$$

and where $p(x)$ is such that $\int_0^a p(x)dx = M$ and $0 \leqq p(x) \leqq H$.

Sharp lower bounds are found in [1] when instead of the condition $p(x) \leqq H$, we have $p(x)$ either monotone, $p(x)$ convex, or $p(x)$ concave. The precise definitions of convex and concave are given below.

In this paper, we find sharp upper bounds for $\lambda_n(p)$ $(n = 1, 2, 3, \cdots)$ whenever $p(x)$ belongs to any one of the following sets of functions:

(a) $E_1(M, H, a)$, the set of monotone increasing functions where

$$\int_0^a p(x)dx = M \text{ and } 0 \leqq p(x) \leqq H,\ x \in [0, a] .$$

(b) $E_2(M, H, a)$, the set of continuous convex functions, i.e., continuous functions $p(x)$ such that

$$p(x) \leqq \frac{x_2 - x}{x_2 - x_1} p(x_1) + \frac{x - x_1}{x_2 - x_1} p(x_2),\ 0 \leqq x_1 \leqq x_2 \leqq a ,$$

with $\int_0^a p(x)dx = M$ and $0 \leqq p(x) \leqq H$, $x \in [0, a]$.

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