ON FUNCTION FAMILIES WITH BOUNDARY

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1. Introduction. Let A be a family of real valued upper semicontinuous functions defined on a compact Hausdorff space E.

A closed set $F \subset E$ is called *determining for* A if every function $f \in A$ attains its maximum on F. If for the space E there exists one and only one minimal determining $F = F(E, A)$ (i.e., a determining set such that no proper closed subset of it is determining), then F is called the *boundary of* E *with respect to the family* A.

A function $h \in A$ is called a *barrier-function of* A at a point $\mathring{x} \in F = F(E, A)$ if and only if $h(\mathring{x}) > h(x)$ for $x \neq \mathring{x}$, $x \in F$.

A point $\mathring{x} \in F$ for which there is a barrier-function of A is called a *semiregular boundary point of* E with respect to A. If for a point $\mathring{x} \in F$ there exists a continuous (at the point $\mathring{x}$) barrier-function, then $\mathring{x}$ will be called a *regular boundary point of* E with respect to A.

Let D be a set contained in a topological space and let $f(x)$ be a real function defined on D. Then the function f^* defined in the closure $\bar{D}$ of D by means of

$$f^*(x) = \limsup_{x' \to x} f(x') \,, \qquad x' \in D, \ x \in \bar{D} \,, \tag{1}$$

is called an *upper regularization of* f.

Let A_1 be a subfamily of A. Then the function

$$\varphi(x) = \{\sup_{f \in A_1} f(x)\}^* \,, \qquad x \in E \,, \tag{2}$$

is called the *upper envelope of* A_1.

Let f be an upper semicontinuous nonnegative function defined in a compact set E. We shall denote by $||f||_E$ the maximum of f on E, $||f||_E = \max_{x \in E} f(x)$.

We say that a family A of functions f defined on E is *separating* (or A *separates the points of* E) if for any two points $x_1 \neq x_2$ of E there is a function $f \in A$ such that $f(x_1) \neq f(x_2)$.

A well known theorem of Šilov [5] asserts: *If A is a family of absolute values of all functions of a separating algebra of complex continuous functions defined on a compact Hausdorff space E, then E has the boundary F with respect to the family A.*

This boundary is sometimes called a *Šilov boundary of* E (with respect to the given algebra).

E. Bishop [3] has recently proved that *if E is metrizable and A is a complete* (with respect to the uniform convergence) *Banach algebra*

Received May 30, 1961. This paper was prepared partially under the sponsorship of NSF Grant 10375.