NONSYMMETRIC PROJECTIONS IN HILBERT SPACE

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0. Introduction. An initial investigation into the kind of operators which can be obtained as the difference of two projections led to the study presented below. In this paper a characterization is given for the general (not necessarily symmetric) bounded linear idempotent operator, or projection, on Hilbert space. These results are applied to the investigation of a projection problem and to a "weak" ordering of such operators. The paper falls naturally into two parts. In the first we give two theorems and several more or less direct consequences which together provide the characterization. In the second part we apply these results to the investigation and solution of a problem which is of importance in probability and statistics. A sketch of the role of this problem in statistical theory and an examination of how our results fit in with previous conclusions complete the present study.

We mention that Dixmier [3] has done work related to the first part, obtaining results of an entirely different nature from ours. So far as we know the point of view presented here does not appear in the literature.

1. Characterization theorems. We utilize the following compressed notation: "positive" for "positive semi-definite", "s.a." for "self adjoint", "skew" for "skew-adjoint". $A \smile B$ indicates that A and B commute, $A \mid \mathscr{V}$ stands for the restriction of the (always linear) operator A to the subspace $\mathscr{V}$, and $\mathscr{R}_A$, $\mathscr{N}_A$ respectively denote the range and the null space of A. The terminology used below is that of complex Hilbert space but as is made clear in the proofs our results apply to the real case as well. In other respects, notation is mostly patterned after that of Riesz and Sz.-Nagy [9].

THEOREM 1. *An operator P on a Hilbert space $\mathfrak{X}$ is a projection (bounded idempotent linear operator) if, and only if, there exist*
(I) a bounded s. a. operator S such that $S^2 - S$ is positive
(II) a unitary operator U, with $U(\overline{\mathscr{R}}_{S^2-S}) \subset \overline{\mathscr{R}}_{S^2-S}$, whose restriction to $\overline{\mathscr{R}}_{S^2-S}$ satisfies

(i) $\quad U^2 = -I$

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