ON AXIOMATIC HOMOLOGY THEORY

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A homology theory will be called *additive* if the homology group of any topological sum of spaces is equal to the direct sum of the homology groups of the individual spaces.

To be more precise let H_* be a homology theory which satisfies the seven axioms of Eilenberg and Steenrod [1]. Let $\mathscr{A}$ be the admissible category on which H_* is defined. Then we require the following.

Additivity Axiom. If X is the disjoint union of open subsets X_α with inclusion maps $i_\alpha: X_\alpha \to X$, all belonging to the category $\mathscr{A}$, then the homomorphisms

$$i_{\alpha *}: H_n(X_\alpha) \to H_n(X)$$

must provide an injective representation of $H_n(X)$ as a direct sum.[1]

Similarly a cohomology theory H^* will be called *additive* if the homomorphisms

$$i_\alpha^*: H^n(X) \to H^n(X_\alpha)$$

provide a projective representation of $H^n(X)$ as a direct product.

It is easily verified that the singular homology and cohomology theories are additive. Also the Čech theories based on infinite coverings are additive. On the other hand James and Whitehead [4] have given examples of homology theories which are not additive.

Let $\mathscr{W}$ denote the category consisting of all pairs (X, A) such that both X and A have the homotopy type of a CW-complex; and all continuous maps between such pairs. (Compare [5].) The main object of this note is to show that there is essentially only one additive homology theory and one additive cohomology theory, with given coefficient group, on the category $\mathscr{W}$.

First consider a sequence $K_1 \subset K_2 \subset K_3 \subset \cdots$ of CW-complexes with union K. Each K_i should be a subcomplex of K. Let H_* be an additive homology theory on the category $\mathscr{W}$.

LEMMA 1. *The homology group $H_q(K)$ is canonically isomorphic to the direct limit of the sequence*

$$H_q(K_1) \to H_q(K_2) \to H_q(K_3) \to \cdots .$$

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[1] This axiom has force only if there are infinitely many X_α. (Compare pg. 33 of Eilenberg-Steenrod.) The corresponding assertion for pairs (X_α, A_α) can easily be proved, making use of the given axiom, together with the "five lemma."