ARITHMETICAL NOTES, III. CERTAIN EQUALLY DISTRIBUTED SETS OF INTEGERS

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1. Introduction. In this note we shall generalize the following two results in the classical theory of numbers. Let n denote a positive integer with distinct prime divisors $p_1, \cdots, p_m$,

$$n = p_1^{e_1} \cdots p_m^{e_m} \quad (m > 0), \quad n = 1\,(m = 0)\,, \tag{1.1}$$

and place $\Omega(n) = e_1 + \cdots + e_m$, $\Omega(1) = 0$, so that $\Omega(n)$ is the total number of prime divisors of n. For real $x \geqq 1$, let $S'(x)$ denote the number of square-free numbers $n \leqq x$ such that $\Omega(n)$ is even, and let $S''(x)$ denote the number of square-free $n \leqq x$ such that $\Omega(n)$ is odd. It is well-known [6, §161] that

$$S'(x) \sim \frac{3x}{\pi^2}, \quad S''(x) \sim \frac{3x}{\pi^2} \qquad \text{as } x \to \infty\,. \tag{1.2}$$

Correspondingly, let $T'(x)$ denote the total number of integers $n \leqq x$ suct that $\Omega(n)$ is even and $T''(x)$ the total number of $n \leqq x$ with $\Omega(n)$ odd. Then [6, §167]

$$T'(x) \sim \frac{x}{2}, \quad T''(x) \sim \frac{x}{2} \quad \text{as } x \to \infty\,. \tag{1.3}$$

The proof of (1.2) is based upon the deep estimate [6, §155] for the Möbius function $\mu(n)$,

$$M(x) \equiv \sum_{n \leqq x} \mu(n) = o(x)\,, \tag{1.4}$$

while the proof of (1.3) is based upon the analogous estimate [6, §167] for Liouville's function $\lambda(n)$,

$$L(x) \equiv \sum_{n \leqq x} \lambda(n) = o(x)\,. \tag{1.5}$$

In Theorem 3.3 we prove a generalization of (1.2) and in Theorem 3.4 the corresponding generalization of (1.3). The respective proofs are based upon an estimate (Theorem 3.1) corresponding to (1.4) for an appropriate extension of $\mu(n)$ and an estimate (Theorem 3.2) corresponding to (1.5) for the analogous extension of $\lambda(n)$. The proofs of these estimates are in the manner of Delange's proofs [3, $I(b), (c)$] of (1.4) and (1.5), both being based upon a classical Tauberian theorem (Lemma 3.2) for the Lambert summabillty process. We also require some elementary

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