A NOTE ON COOK'S WAVE-MATRIX THEOREM

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1. Introduction. Consider the linear operator H_0 defined by

$$[H_0u](\vec{x}) = -\nabla^2 u(\vec{x}) + V(\vec{x})u(\vec{x}) \tag{1.1}$$

over all $\vec{x} \in R_n$, n-dimensional Euclidean space, for each $u \in \mathscr{D}_0$. Here ∇^2 is the Laplacian and we take $\mathscr{D}_0$ as the set of all complex valued functions u over R_n which everywhere possess continuous partials of all orders $\leqq 2$ and which together with these partials are in absolute value $\leqq Q(|\vec{x}|)\exp(-2^{-1}|\vec{x}|^2)$ over R_n for some polynomial Q depending on u. Here V is a fixed, real valued, measurable function over R_n subject to additional assumptions below which will assure that H_0 takes $\mathscr{D}_0$ into $X = L_2(R_n)$ as a symmetric operator in the Hilbert space X.

Assuming that $V \in L_2(R_n)$ for $n = 3$, Cook [2] has shown that the unique existent (see Theorem I following) self-adjoint extension H of H_0 has the unitary operator

$$W(t) = e^{itH}e^{-it\tilde{H}}, \tag{1.2}$$

where $\tilde{H}$ is the similar extension of $\tilde{H}_0$ and $\tilde{H}_0$ differs from H_0 only by replacing $V(\vec{x})$ by zero in (1.1), to have existent isometric operators $W_\pm$ on X which are the strong limits of $W(t)$ as $t \to \pm\infty$. Moreover, $W_\pm \tilde{H} = HW_\pm$, the range spaces $Y_\pm = W_\pm X$ reduce H, and each H eigenvector is orthogonal to $Y_\pm$. In Theorem II bellow we give a significant sharpening of these results by weakening the restrictions upon V at ∞. Thus, with arbitrary $\rho > 0$, any function of the form $C|\vec{x}|^{-1-\rho}$ over $|\vec{x}| \geqq b$ will qualify under our assumptions (the Coulomb case $C|\vec{x}|^{-1}$ thus being borderline), while only such of form $C|\vec{x}|^{-3/2-\rho}$ there will do so under Cook's assumptions. In Theorem III we also generalize to dimension $n \geqq 3$. Cook's results are used by Ikebe [4] in showing $S = W_+^* W_-$, the "S-matrix", to be unitary with $Y_+ = Y_-$ and in showing the expected connection of the positive part of the spectrum of H with scattering theory under considerably more stringent conditions upon V. Our $n = 3$ existence result II for $W_\pm$ also includes that of Jauch & Zinnes ([5], p. 566), who assume $V(\vec{x}) = C|\vec{x}|^{-\beta}$ with $1 < \beta < 3/2$, and that of Hack [3], who replaces ${}_+\|V\|_\gamma < +\infty$ for some $\gamma \in [2,3)$ by the above noted stronger assumption that $|V(\vec{x})| \leqq M|\vec{x}|^{-1-\rho}$ over $|\vec{x}| \geqq b$ for some $\rho > 0$.*

2. Statements. As notation for our theorems, denote $D_b^+ = \{\vec{x} \in R_n \mid |\vec{x}| \geqq b\}$

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* *Note added in proof.* See also Kuroda, Nuovo Cim., **12**, (1959), p. 431-454 particularly Theorem 4.1), p. 444.