ITERATIONS OF GENERALIZED EULER FUNCTIONS

G. K. White

1. Introduction. In this paper p and q will denote primes. We recall that a function $f(n)$ of an integral variable $n \geqq 1$ is said to be multiplicative, if

$$f(mn) = f(m)f(n) \tag{1}$$

whenever $(m, n) = 1$, and additive, if

$$f(mn) = f(m) + f(n) \tag{2}$$

whenever $(m, n) = 1$. If however $f(n)$ satisfies (2) for all integers $m \geqq 1$, $n \geqq 1$ we shall say that $f(n)$ is *completely additive*. Consider a multiplicative integral-valued function $\psi(n) > 0$ and put

$$\psi_0(n) = n,\ \psi_1(n) = \psi(n),\ \cdots,\ \psi_r(n) = \psi[\psi_{r-1}(n)],\ \cdots . \tag{3}$$

We shall say that $\psi(n)$ is of finite index if, to each $n > 1$, there is an integer $C = C(n)$ such that

$$\psi_r(n) \begin{cases} > 1 \text{ for } r \leqq C \\ = 1 \text{ for } r > C , \end{cases} \tag{4}$$

in which case we put $C(1) = 0$.

The familiar Euler function

$$\varphi(n) = \sum_{\substack{m \leqq n \\ (m,n)=1}} 1 = n \prod_{p/n} \left(1 - \frac{1}{p}\right) \tag{5}$$

is an example of such a function, since $\varphi(n) < n$. For this case ($\psi = \varphi$), properties of the corresponding function $C(n)$ were investigated by Pillai [1], who attributes the problem to Vaidyanathaswami. Later, Shapiro [2, 3, 4] observed that this particular $C(n)$ satisfied the condition

$$C(mn) = C(m) + C(n) + \begin{cases} 1 \text{ for } m, n \text{ both even} \\ 0 \text{ otherwise} , \end{cases} \tag{6}$$

and went on to obtain, inter alia, a certain class (S) of multiplicative functions $\psi(n)$ of finite index satisfying (6). In a restricted sense, (S) consists of functions similar in form to $\varphi(n)$; for example they satisfy

$$\psi(x^n)[\psi(x)]^{n-2} = [\psi(x^2)]^{n-1}$$

for all positive integers x, n.

Received June 15, 1961.