ON THE PROJECTIVE COVER OF A MODULE AND RELATED RESULTS

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Introduction. The concept of the "injective envelope" of a module was first given by Eckmann and Schopf [2], although this terminology was first employed by Matlis [5]. The injective envelope of a module always exists and is unique in a certain sense. The dual concept of the "projective cover" of a module has been given by Bass [1], and the concept of "minimal epimorphism" in a "perfect category" as defined by Eilenberg [3] is a particular case of this concept. The projective cover of a module does not always exist but is unique whenever it exists. Eilenberg [3] has proved that every module in a perfect category possesses a projective cover. Bass [1] calls a ring "perfect" if every module over the ring possesses a projective cover, and he gives several characterizations of a perfect ring. We shall call a module "perfect" if it possesses a projective cover. It would be natural to try to characterize a perfect module, but it seems likely that such attempts may result in obtaining equivalent definitions of the projective cover of a module. One might instead consider specific types of modules and try to obtain necessary and sufficient conditions so that they may be perfect. In §1 we first define a category of perfect modules and then give a necessary and sufficient condition for a finitely generated module over a Noetherian ring to be perfect. In §2 we give some results on "essential monomorphism" and "minimal epimorphism" [1, 2]. In §3 we give some results on modules over perfect rings. In §4 we give new proofs of some known results to show how the concepts of the injective envelope and the projective cover of a module simplify the proofs considerably.

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1. A category of perfect modules. Let Λ be a ring with unit element $1 \neq 0$. Throughout this paper we shall be concerned with unitary left Λ-modules and so we shall call them simply modules. We recall some definitions. Let $f\colon L \rightarrow M$ be a homomorphism of modules. If $H \cap Im\ f = 0$ implies $H = 0$, where H is a submodule of M, f is called an *essential* homomorphism; moreover, if f is a monomorphism and M is an injective module, then M is called the *injective envelope* of L and is denoted by $E(L)$. If, however, $K + \text{Ker}\ f = L$ implies $K = L$, where

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