DERIVATIVES OF THE HARMONIC MEASURES IN MULTIPLY-CONNECTED DOMAINS

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1. Introduction. Basic definitions and some known results. The Geometric Function Theory encounters serious difficulties when dealing with multilply-connected domains, due to the fact that no domain function is known in an explicit form for a domain of connectivity greater than two. It is for this reason that one at least tries to find properties of the domain functions in terms of geometric characteristics of their domains.

In this paper we search for information on domain functions and certain families of functions, defined for p-connected domains, that can be deduced form properties of functions defined in *symmetric* domains of connectivity $2(p-1)$. We also extend the results to infinitely-connected domains.

Let Δ be a domain in the z-plane, and let $L^2(\Delta)$ be the class of functions which are regular and square integrable in Δ. Let $l^2(\Delta)$ be its subclass consisting of those functions which have a single-valued integral in Δ. Both classes form separable Hilbert Spaces $[L^2(\Delta)]$ and $[l^2(\Delta)]$ under the scalar multiplication

$$(1.1) \qquad (f; \bar{g}) = \iint_\Delta f(z)\,\overline{g(z)}\,d\omega\,, \qquad d\omega = dx\,dy,\; z = x + iy\,.$$

Let $h^2(\Delta)$ be the class of functions which belong to the orthogonal complement of $[l^2(\Delta)]$ with respect to $[L^2(\Delta)]$. The Hilbert space $[h^2(\Delta)]$ has a finite dimension $p-1$, if Δ is p-connected and none of its boundary components reduces to a point. (See Bergman [3]). If Δ is infinitely-connected, this space has in general an infinite dimension (See Virtanen [9]).

According to Virtanen [9] (See also Nevanlinna [7], one can construct an orthogonal basis for $h^2(\Delta)$ as follows: Let $C_1, C_2, \cdots$ be a homology basis of cycles in Δ, subject therefore to the following conditions:

(1) Any cycle in Δ is homologous to a finite chain of these cycles.

(2) No chain is homologous to zero, unless its coefficients are all zero.

We can also assume that each cycle C_j, $(j = 1, 2, \cdots)$[1] is an oriented analytic Jordan curve. If $K_\Delta(z, \zeta)$ is the Bergman kernel function for the class $L^2(\Delta)$, then the functions

$$(1.2) \qquad F_j'(z; \Delta) = i \oint_{C_j} \overline{K_\Delta(\zeta, z)}\,\overline{d\zeta}\,, \qquad j = 1, 2, \cdots\,.$$

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[1] Throughout this paper the notation $1, 2, \cdots$ will mean a finite or infinite sequence, as the case may be.