GROUP MEMBERSHIP IN SEMIGROUPS

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An element a of a semigroup S is said to be a *group member* in S if there is a subgroup G of S which contains a. This notion was introduced in 1909 by Arthur Ranum in discussing group membership in certain sets of matrices. In 1927 he extended the notion to group membership in finite dimensional algebras. Using somewhat different methods, the subject was revived separately by two sets of authors, W. E. Barnes and H. Schneider in 1957, and H. K. Farahat and L. Mirsky in 1956 and 1958. They discussed the notion of group membership in polynomials in an element algebraic over a field and in rings of various types. With regard to rings, we say that an element a is a group member in a ring R if a is a group member of the semigroup formed by elements of R and its multiplicative operation. Barnes and Schneider posed the following question: If S is a subring of a ring R and an element a of S is a group member in R, under what conditions is it a group member in S [1, p. 168]? Farahat and Mirsky echo this question, in a note added in proof to their 1958 article, and state that one of their theorems gives a "partial answer" to this question [3, p. 244]. We propose to return to another of Ranum's notions, that of "associated group member," and develop the theory for semigroups. We shall see that this will lead to a more complete solution of the question proposed, and one which applies equally well to semigroups in general.

It is known [3, p. 232] that if a is a group member in a semigroup S, then there is a maximal subgroup $M(a)$ of S, containing a, whose identity is the identity of every subgroup containing a. Further, the inverse of a, relative to this identity, is the same in every subgroup containing a. Distinct maximal subgroups are disjoint. We say that n is the *group index* of a in S if a^n is a group member in S and n is the smallest positive integer for which this is true. (Thus the group index of a group member is 1.) The following theorem is perhaps not so well known.

THEOREM 1. *If S is a semigroup and n is the group index of an element a in S, then a^t is a group member in S if and only if $t \geqq n$. Furthermore, if $t \geqq n$, then $M(a^t) = M(a^n)$.*

Proof. Let a^t be a group member in S. By the definition of group index, t cannot be less than n. Hence $t \geqq n$.

Conversely, let $t \geqq n$. If $t = n$, then $a^t = a^n$ and so a^t is a group

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