ABSOLUTE CONTINUITY OF INFINITELY DIVISIBLE DISTRIBUTIONS

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1. Introduction and summary. A probability distribution function F is said to be infinitely divisible if and only if for every integer n there is a distribution function F_n whose n-fold convolution is F. If F is infinitely divisible, its characteristic function f is necessarily of the form

$$(1) \qquad f(u) = \exp\left\{iu\gamma + \int_{-\infty}^{\infty}\left(e^{iux} - 1 - \frac{iux}{1+x^2}\right)\frac{1+x^2}{x^2}dG(x)\right\},$$

where $u \in (-\infty, \infty)$, γ is some constant, and G is a bounded, non-decreasing function. J. R. Blum and M. Rosenblatt [1] have found necessary and sufficient conditions that F be continuous and necessary and sufficient conditions that F be discrete. The purpose of this note is to add to the results of Blum and Rosenblatt by giving sufficient conditions under which an infinitely divisible probability distribution F is absolutely continuous. These conditions are that G be discontinuous at 0 or that $\int_{-\infty}^{\infty}(1/x^2)dG_{ac}(x) = \infty$, where G_{ac} is the absolutely continuous component of G. In § 2 some lemmas will be proved, and in § 3 the proof of the sufficiency of these conditions will be given. All notation used here is standard and may be found, for example, in Loève [2].

2. Some lemmas. In this section three lemmas are proved which will be used in the following section.

LEMMA 1. *If F and H are probability distribution functions, and if F is absolutely continuous, then the convolution of F and H, $F * H$, is absolutely continuous.*

This lemma is well known, and the proof is omitted.

LEMMA 2. *If $\{F_n\}$ is a sequence of absolutely continuous distribution functions, and if $p_n \geqq 1$ and $\sum_{n=1}^{\infty} p_n = 1$, then $\sum_{n=1}^{\infty} p_n F_n$ is an absolutely continuous distribution function.*

Proof. By using the Lebesgue monotone convergence theorem it is easy to verify that $\sum_{n=1}^{\infty} p_n f_n$ is the density of $\sum_{n=1}^{\infty} p_n F_n$, where f_n is the density of F_n.

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