OPERATORS OF FINITE RANK IN A REFLEXIVE BANACH SPACE

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1. Let X be a reflexive Banach space and $F(X)$ the Banach algebra of all uniform limits of operators of finite rank, in X. Bonsall [1] has characterized $F(X)$ as a simple, $B^\#$-annihilator algebra: $F(X)$ contains no proper closed two-sided ideals, every proper, closed right (left) ideal of $F(X)$ has a nonzero left (right) annihilator, and, given any $T \in F(X)$, there exists $T^\# \in F(X)$ such that

$$||T|| \, ||T^\#|| = ||(TT^\#)^n||^{1/n} , \qquad n = 1, 2, 3, \cdots .$$

In this note, we obtain a new characterization for $F(X)$ (Theorem 3.2): a Banach algebra A is the algebra $F(X)$ of all uniform limits of operators of finite rank in a reflexive Banach space X if and only if A is a simple, weakly compact, $B^\#$-algebra with minimal ideals (A is weakly compact if left- and right-multiplications by every $a \in A$ are weakly compact operators). In the process of proving this result, we obtain a characterization of reflexive Banach spaces which seems to be of some independent interest (Theorem 2.2): a Banach space X is reflexive if and only if every operator in X of rank 1 is a weakly compact element of $B(X)$.

2. Let X be a Banach space and $B = B(X)$ the Banach algebra of all bounded operators in X with the uniform topology. For $T \in B$, let R_T denote the operator in B obtained by multiplying elements of B on the right by T: $R_T(A) = AT$ for $A \in B$.

Suppose that T is a fixed operator of rank 1 in X with $H = [x \in X: Tx = 0]$. Then H is a closed hyperplane in X and if x_0 is an element of X such that $Tx_0 \neq 0$, then $X = H \oplus (x_0)$ and we may assume that $||x_0|| = 1$. Write $B' = [S \in B: S(H) = (0)]$. For each $S \in B'$, we define an element x_S of X by setting $x_S = S(x_0)$. The mapping $S \rightarrow x_S$ is clearly linear.

LEMMA 2.1. *The linear mapping* $S \rightarrow x_S$ *is a homeomorphism of* B' *onto* X.

Proof. It is clear that the mapping is one-to-one and, since $||S(x_0)|| \leqq ||S||$, it is continuous. It is also onto; in fact, let $\varphi \in X^*$ be such that $\varphi(H) = (0)$, $\varphi(x_0) = 1$. Then for given $x \in X$, the operator S_x defined by setting $S_x(y) = \varphi(y)x$, $y \in X$ belongs to B' and is mapped into x by the mapping $S \rightarrow S(x_0)$. Hence, by the closed graph theorem, the

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